Transverse Unknotting Number and Family Trees

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Abstract

The unknotting number is a classical invariant for smooth knots, [1]. More recently, the concept of knot ancestry has been defined and explored, [3]. In my research, I explore how these concepts can be adapted to study transverse knots, which are smooth knots that satisfy an additional geometric condition imposed by a contact structure.

1. Introduction

Smooth knots are well-studied objects in topology. A smooth knot is a closed curve in 3-dimensional space that does not intersect itself anywhere. Figure 1 shows a diagram of the unknot and a diagram of the positive trefoil knot.

FIGURE 1. A diagram of the unknot and a diagram of the positive trefoil knot.

Two knots are equivalent if one can be deformed to the other. It is well-known that two diagrams represent the same smooth knot if and only if their diagrams are equivalent through Reidemeister moves. On the other hand, to show that two knots are different, we need to construct an invariant that can distinguish them. For example, tricolorability shows that the trefoil is different from the unknot. Unknotting number is another invariant: it is known that every smooth knot diagram can be converted to the diagram of the unknot by changing crossings. This is used to define the smooth unknotting number, which measures the minimal number of times a knot must cross through itself in order to become the unknot.

This thesis will focus on transverse knots, which are smooth knots that satisfy an additional geometric condition imposed by a contact structure. Figure 2 shows transverse representatives of the unknot.

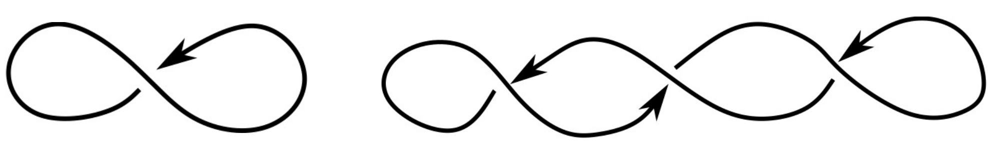


FIGURE 2. Transverse representatives of the unknot.

Briefly, a contact structure is given by a field of planes on $\mathbb{R}^3$, and transverse knots will be oriented and have tangent vectors that are always transverse to these planes. More background on contact structures can be found in Section 3. We observe transverse knots in their front diagrams, which are shown in the xz-plane with the missing positive y-axis pointing into the page. The transversal condition creates forbidden vertical tangencies and crossing in front diagrams as shown in Figure 3, to be discussed further in Section 3.

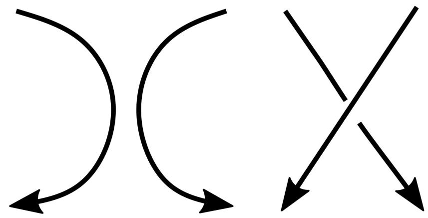


Figure 3. Downward vertical tangencies and down-down positive crossing are forbidden.

Two transverse knots are equivalent if one can be deformed to the other without introducing any forbidden segments under the transversal condition in the process of deformation. Just as there are Reidemeister moves for smooth knots, there are analogous Reidemeister moves for transverse knots as long as they do not introduce forbidden crossing or tangenices. Two transverse knots are transversely isotopic if and only if their diagrams are equivalent under transverse Reidemeister moves.

Just as smooth knots have invariants that distinguish different smooth knots, there are invariants for transverse knots. The two classical transverse knot invariants are: topological knot type (smooth knot type) and the *self-linking number.* The self-linking number is the writhe of the knot in the front diagram, defined by subtracting the number of negative crossings from the number of positive crossings. The self-linking number shows that the two transverse unknots in Figure 2 are different transverse unknots, and in fact, Lemma 3.16 shows us that there are an infinite number of transverse unknots.

There is a special operation called stabilization that allows us to move between the two different transverse unknots in Figure 2. Stabilization of a transverse knot is formed by taking an arc in the front diagram and doing a "double twist." Performing a stabilization on a transverse knot decreases the self-linking number by two, as two negative crossings are added in the operation. For transversely simple knot types like the unknot, any two transverse representatives are related by some number of stabilizations.

We study the notion of unknotting for transverse knots. In contrast to the smooth setting, in a transverse setting only certain crossings can be changed. We call crossing changes that are allowable in front diagrams of transverse knots *transverse crossing changes.* Figure 4 shows the transverse crossing changes.

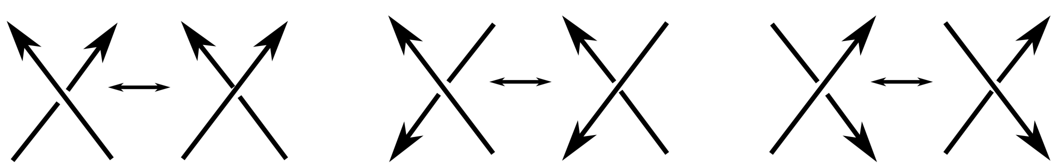


Figure 4. Transverse crossing changes.

In this thesis, we define a new invariant: the transverse unknotting number. In Theorem 5.1, we prove that every smooth knot has a transverse representative that can be transversely unknotted. Building on this theorem in Section 5, we have the following important result about transversely unknotting transverse knots:

Theorem 1.1. *Every transverse knot can be transversely unknotted.*

Because there are many different transverse unknots, we could define the transverse unknotting number as the minimum number of transverse crossing changes necessary to reach any transverse unknot or a particular transverse unknot. In this thesis, we define transverse unknotting number by measuring how far away we are from the transverse unknot with maximum self-linking number, $sl = -1$.

In Lemma 6.3, we establish a lower bound for the transverse unknotting number of a transverse knot T, which is the maximum of the unknotting number of its smooth knot type and the number of transverse crossing changes needed to reach $sl(T) = -1$:

$$\max\left\{u(S),\ \frac{sl(T)+1}{2}\right\} \leq U^{\pitchfork}_{-1}(T).$$

Furthermore, because a stabilization can be done and undone through a transverse crossing change as shown in Lemma 5.3, we prove that $S^n(T)$, an n-th stabilization of a transverse knot T, has an upper bound to its transverse unknotting number: $U^{\pitchfork}_{-1}(S^n(T)) \leq U^{\pitchfork}_{-1}(T)+n$. Combining this upper bound with Lemma 6.3, we then have the following corollary:

Corollary 1.2. For a transversely simple smooth knot type S with maximum self-linking number $m \leq -3$, assume that the maximum self-linking number transverse representative T has transverse unknotting number $U^{\pitchfork}_{-1}(T) = |{}^{(sl(T)+1)}/_{2}|$. Then, it follows that for any transverse knot T_i in this smooth knot type S, the transverse unknotting number is

$$U^{\pitchfork}_{-1}(T_i) = \frac{sl(T_i)+1}{2}\ .$$

This thesis also explores how the ancestor-descendant relationship in smooth knots can be applied to study transverse knots. As defined by Cantarella, Henrich, Magness, O'Keefe, Perez, Rawdon, and Zimmer in [3], a smooth knot K_1 is an ancestor of K_2 if K_2 can be obtained from a minimal crossing diagram of K_1 by some number of crossing changes. Note that without the minimal crossing condition in the definition, the ancestor-descendant relationship can be defined for any two smooth knots. Refer to Lemma 2.10 for further information on why the minimal crossing diagram is necessary.

Cantarella et al. also showed that twist knots and $(2,p)$-torus knots are two knot families that are closely related to each other in terms of ancestor-descendant relationship. They proved that a knot K is a descendant of a twist knot T_n if and only if $K = T_k$ for some integer k with $0 \leq k \leq n$. They also proved that a knot K is a descendant of a torus knot $T_{2,p}$ if and only if $K = T_{2,q}$ for some integer $0 < q \leq p$, where p and q are odd integers.

As an adaptation of this ancestor-descendant relationship, we define the *transverse family tree*. A sequence of transverse knots $(T_1, T_2, \ldots, T_n)$ is a transverse family tree if each T_{i+1} can be obtained from T_i by a single transverse crossing change. Furthermore, we define that a transverse family is *maximal* if each T_i has maximal self-linking number in its knot type. Lastly, a transverse family tree is *increasing* if the self-linking numbers of T_i are strictly increasing and *decreasing* if the self-linking numbers are strictly decreasing.

Lemma 7.3 shows that given any smooth knot types K_1 and K_2, there exists a transverse family tree $(T_1, \ldots, T_2)$ where T_1 is the maximum self-linking number transverse representative of knot type K_1 and T_2 is of K_2. Refer to Section 7 for more information on transverse family trees and the proof of Lemma 7.3.

Regarding transverse family trees, we have the following major results about twist knots and $(2,p)$-torus knots:

Theorem 1.3 (Twist Knot Transverse Family Trees).

(1) *For any odd $m \geq 1$, there exists a maximal decreasing transverse family tree $(T_m, T_{m-2}, \ldots, T_1)$, where T_j is a transverse representative of the twist knot K_j.*
(2) *For any even $m \geq 2$, there exists a maximal decreasing transverse family tree $(T_m, T_{m-2}, \ldots, T_2)$, where T_j is a transverse representative of the twist knot K_j.*
(3) *There exists a transverse family tree $(T_m, U, T_{m-1}, U, \ldots, T_1)$, where T_j is a transverse representative of the twist knot K_j and U is a transverse unknot.*

Theorem 1.4 (Torus Knot Transverse Family Trees).

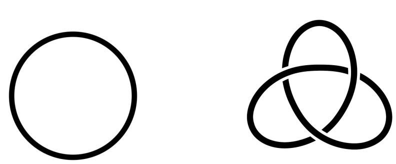
(1) *For all odd $p \geq 3$, there exists a maximal decreasing transverse family tree $(T_{2,p}, T_{2,p-2}, \ldots, T_{2,3})$, where $T_{2,j}$ is a transverse representative of the torus knot $K_{2,j}$.*
(2) *For all odd $n \leq -3$, there exists a maximal increasing transverse family tree $(T_{2,n}, T_{2,n+2}, \ldots, T_{2,-3})$, where $T_{2,j}$ is a transverse representative of torus knot $K_{2,j}$.*

Section 2 provides a general background about smooth knots and introduces the newest concept of the ancestor-descendant relationship as defined by Cantarella et al. Section 3 introduces transverse knots, which are the primary objects of study in this paper. The section includes an explanation of the forbidden segments in front diagrams of transverse knots, a proof that every smooth knot has a transverse representative, and the definition of stabilization. Section 4 discusses crossing changes in transverse knots, which are realized differently from smooth knots due to the forbidden crossing. Section 5 explores transverse crossing changes further and addresses the question of whether all transverse knots can be transversely unknotted. It includes a proof that every smooth knot has a transverse representative that can be unknotted and contemplates the relationship between a stabilization and a transverse crossing change to answer this question. Section 6 defines the transverse unknotting number, which measures how far away a transverse knot is from the $sl = -1$ transverse unknot. It also includes lemmas about lower and upper bounds of the transverse unknotting number. The last section examines the new notion of a transverse family tree, which is an adaption of ancestor-descendant relationships in smooth knots. It includes major results about twist knot transverse family trees and $(2, p)$-torus knot transverse family trees.

2. Smooth Knots

In this section, we will review some known definitions and results about smooth knots. It is these concepts and results that we will try to extend to the world of transverse knots.

A **knot** is a closed curve in space that does not intersect itself anywhere. The simplest knot of all knots is the unknotted circle, which is called the unknot or the trivial knot.



(A) An unknot. (B) A trefoil.

Figure 5. Examples of smooth knots.

The next simplest knot is called a trefoil knot, as shown in Figure 5 (B). There are many different pictures of the same knot, and such a picture is called a **diagram** of the knot.

A **diagram** is the knot's projection together with a broken segment at each crossing that indicates the strand with lower height.

Two knots K_1 and K_2 are *isotopic* (or *equivalent*) if we can deform K_1 to K_2 without breaking and re-gluing the strands. More formally:

Definition 2.1. Two knots K and $\widetilde{K}$ are **isotopic** if there is a family of knots K_t, parameterized by $t \in [0, 1]$, such that $K_0 = K$ and $K_1 = \widetilde{K}$.

In fact, two knots are isotopic if and only if their diagrams are equivalent via *planar isotopy* and *Reidemeister moves*.

Definition 2.2. A deformation of a knot diagram is called a **planar isotopy** if it can be obtained from a deformation of the diagram plane as if it were made of rubber with the diagram drawn upon it.

Definition 2.3. A **Reidemeister move** is one of three ways to change a diagram of the knot that *will* change the number of or the relation between crossings.

The **first Reidemeister move** allows us to put in or take out a twist in the knot. The **second Reidemeister move** allows us to either add two crossings or remove two crossings. The **third Reidemeister move** allows us to slide a strand of the knot from one side of a crossing to the other side of the crossing.

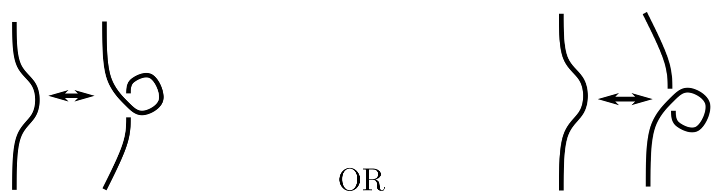


Figure 6. Type I Reidemeister moves.

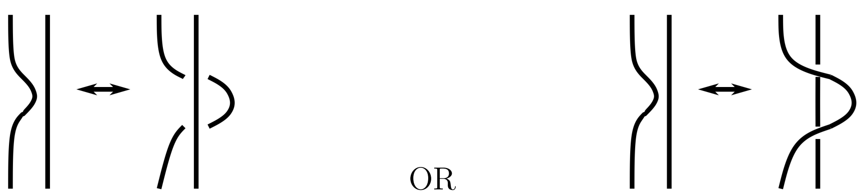


Figure 7. Type II Reidemeister moves.

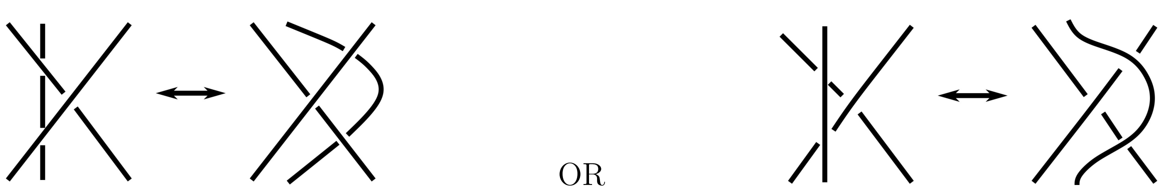


Figure 8. Type III Reidemeister moves.

In 1926, mathematician Kurt Reidemeister proved:

Theorem 2.4. *Two knot diagrams represent equivalent knots if and only if the diagrams are equivalent under planar isotopy and Reidemeister moves.*

While we can use planar isotopy and Reidemeister moves to show that two knots are equivalent, we use something called a *knot invariant* to show that two knots are different.

Definition 2.5. A **knot invariant** is a quantity defined for each knot that is the same for equivalent knots.

Examples of knot invariants include tricolorability, unknotting number, bridge number, crossing number, etc. While knot invariants can be used to distinguish different knots, having the same knot invariant quantity does not necessarily imply that the two diagrams are the same knot. For the purpose of this research, we will focus on unknotting number. To define unknotting number, we first need to look at the following theorem.

Theorem 2.6. *Given any knot diagram, by changing crossings, it becomes the diagram of the unknot.*

The following proof is from Adams' *The Knot Book*, [1].

Proof. Given a diagram of a knot, select a starting point on the diagram that, for the sake of convenience, is not at a crossing, and select a direction to traverse the knot. Starting at that point, head along the knot in the chosen direction. Whenever we arrive at a particular crossing, change the crossing if necessary so that the strand that we are on is the overstrand. If we come to a crossing that we have already been through once, do not change the crossing, but continue through it on the understrand. Once we return to our initial starting point, we have a diagram of a knot that we obtained from the original knot diagram by changing crossings. This new diagram will in fact be the trivial knot.

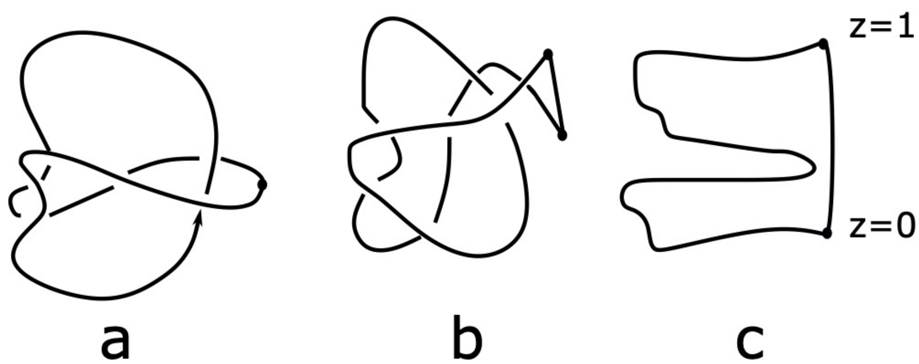


FIGURE 9. (a) Altered diagram. (b) Partial side view of the diagram. (c) Side view of the diagram.

To see that this diagram actually is the trivial knot, view it in three-space. Think of the z axis as coming straight out of the diagram towards us. Starting at the initial point again, place that point in three-space with z-coordinate $z = 1$. As we traverse the knot, decrease the z-coordinate of each of the points on the knot until we almost reach the initial starting point. The last point will have z-coordinate $z = 0$. But since the last point and the initial starting point should be the same point, we must put in a vertical bar from one to the other to complete the knot. Note then that when we look straight down the z axis at our knot, we see the diagram that we had changed the crossings to create. But when we look at our diagram from the side, we see a diagram with *no crossings*. Hence, this new knot diagram is actually a trivial knot. □

From this theorem, we can define the knot invariant *unknotting number*. Because every diagram of a knot can be turned into a diagram of the unknot by changing some subset of the crossings in the diagram, it follows that every knot has a finite unknotting number.

Definition 2.7. A knot K has **unknotting number** n if there exists a diagram of the knot such that changing n crossings in the diagram turns the knot into the unknot and there is no diagram such that fewer changes would have turned it into the unknot.

In this definition of the unknotting number, we perform all the crossing changes in a single diagram of the knot. According to Adams in *The Knot Book*, the unknotting number is traditionally defined to be the least number of crossing changes necessary to change a knot into an unknot, where we can perform the first crossing change in one diagram of the knot, then do an ambient isotopy of the resulting diagram to a new diagram and change the second crossing in that diagram. We can then do another ambient isotopy to a new diagram before we change our third crossing, and continue in this manner until we have done all n crossing changes.

The two definitions are in fact equivalent, as described in [1]. We can keep track of each crossing change in the second definition with an arc that runs to and from the two points on the knot where the crossing change occurs. As we do our ambient isotopy to another diagram, we carry along these arcs, stretching and deforming them as necessary. By the time we finish our n crossing changes, we have n such arcs. However, we can then shrink each of these arcs down to a tiny arc, pulling the knot along, and make a single diagram of the knot so that each arc appears as a vertical arc running from top of a crossing to the bottom. Then, changing these crossing in this single diagram is equivalent to changing the crossings one by one and allowing ambient isotopy to occur between the crossing changes.

Example 2.8. To understand the definition of unknotting number, consider the trefoil as an example. We know that the trefoil is not equivalent to the unknot because the trefoil is tricolorable[1] while the unknot is not. Figure 10 illustrates this.

(A) A trefoil. (B) An unknot.

FIGURE 10. The trefoil is tricolorable while the unknot is not.

Because the trefoil is not equivalent to the unknot, we know that the unknotting number of the trefoil must be at least 1. We will now prove that the unknotting number of the trefoil

[1] We say that a knot is **tricolorable** if each of the strands in the diagram can be colored one of three different colors, such that at each crossing either three different colors come together or all the same color comes together and at least two colors are used, [1]. Because all Reidemeister moves preserve tricolorability, tricolorability is a knot invariant. This means that if a diagram is tricolorable, then all other diagrams of the same knot are tricolorable as well.

is in fact 1 by showing that the trefoil diagram in Figure 5 (B) can be unknotted with just a single crossing change.

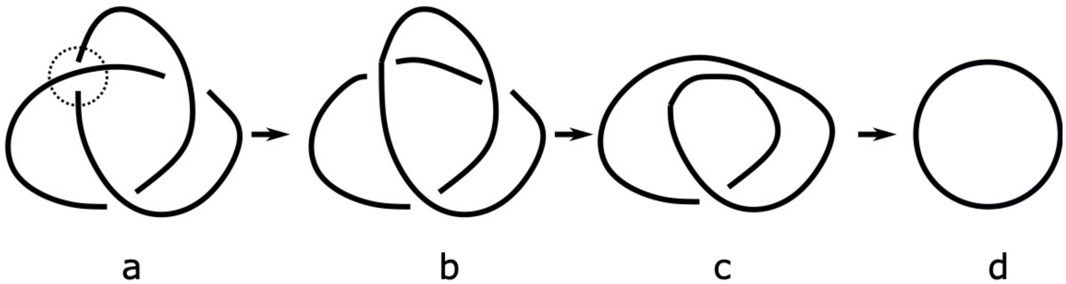


FIGURE 11. Unknotting the trefoil.

Figure 11 (a) shows which crossing we will change in the diagram of the trefoil. Figure 11 (b) shows the diagram after one crossing change is made. Then, we use Type II Reidemeister move to reach the diagram in (c). We finally use Type I Reidemeister move to reach the diagram in (d), which is clearly a diagram of the unknot.

Hence, we have shown that there exists a diagram of the trefoil that can be unknotted with just one crossing change. By the definition of unknotting number, we then conclude that the trefoil has unknotting number 1.

As we can see in the above example, it is simple to calculate the unknotting number if there exists a diagram that can be unknotted with just one crossing change. However, in general, the unknotting number of a knot is very difficult to calculate, because it is hard to verify that the number in question really is the minimal number of crossing changes necessary to unknot the knot across all its diagrams. Moreover, sometimes the unknotting number is not realized in a minimal crossing diagram of the knot.

It is known that the unknotting number of a non-trivial twist knot is 1, and the unknotting number of a $(p, q)-$torus knot is ${}^{(p-1)(q-1)}\!/_2$. The unknotting number of all prime knots with 9 or fewer crossings have all been determined and can be found on the *Knotinfo* website, [14], along with information about other knot invariants.

While the unknotting number is a classical invariant, recently the new concept of ancestor-descendant relationship has been defined and explored by Cantarella, Henrich, Magness, O'Keefe, Perez, Rawdon, and Zimmer, [3].

Definition 2.9. A knot H is a **descendant** of another knot K if H can be obtained from a *minimal* crossing diagram of K by some number of crossing changes. In this case, we say that K is a **ancestor**[2] of H.

Note the difference in definition between unknotting number and ancestor-descendant relationship. Unknotting number focuses on reaching the unknot through crossing changes, while ancestor-descendant relationship examines obtaining another knot (not necessarily the unknot) from a minimal crossing diagram of a knot through crossing changes.

[2]Cantarella, Henrich, Magness, O'Keefe, Perez, K., Rawdon, and Zimmer defined K in this relationship as a **parent** of H. However, the term **ancestor** seems more appropriate here as a complement to a descendant, as **parent** suggests a closer relationship. For the purpose of this paper, we will continue to define this knot relationship as an ancestor-descendant relationship.

It is important to note that by definition the descendant knot is obtained from a *minimal* crossing diagram of the ancestor knot. The following lemma shows that without this minimal crossing condition, the ancestor-descendant relationship can be defined for any two knots.

Lemma 2.10. Given any two knots K_1 and K_2, there exists a not necessarily a minimal crossing diagram of K_1 that can be changed to a diagram of K_2 by crossing changes.

Proof. Let K_1, K_2 be any two knots that are not equivalent to each other. Consider the following knot shadow[3] of the composition knot $K_1 \# K_2$.

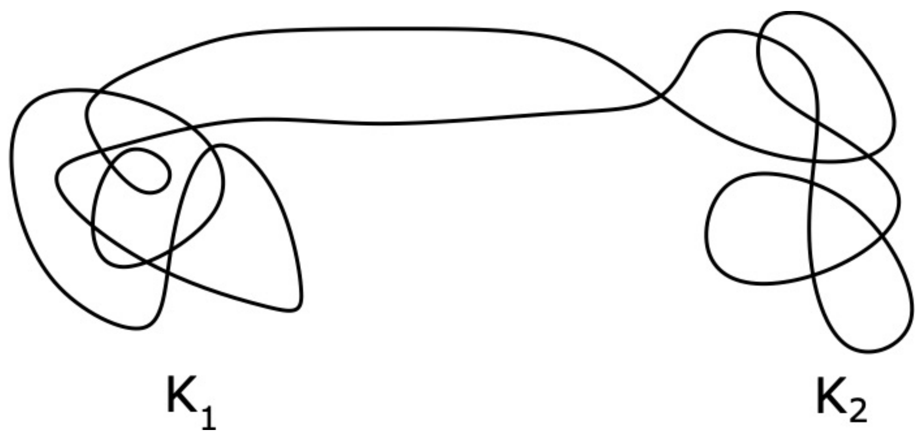


FIGURE 12. A knot shadow of the composition knot $K_1 \# K_2$

We want to prove that there exists a diagram of K_1 that can be changed to a diagram of K_2 by crossing changes. First, choose crossings in the K_1 section of Figure 12 so that we have the K_1 knot on the left side. By Theorem 2.6, we can choose crossings in the K_2 section of Figure 12 so that we have an unknot on the right side. Note that this unknot will have the same knot shadow as the diagram of K_2 in the original composition knot.

We now have a diagram of the K_1 that can be changed to a diagram of K_2 by crossing changes. We simply reverse what we did on both sides. On the left side where we have the K_1 knot, we go through the process of crossing changes described in Theorem 2.6 to turn it into an unknot. On the right side where we have the unknot, we can change crossings so that we now have the K_2 knot. This is possible because the unknot was originally produced from the knot shadow of K_2.

Thus, we have shown that there exists a diagram of K_1 that is not necessarily a minimal crossing diagram that can be changed to a diagram of K_2 by crossing changes. Similarly, there exists a diagram of K_2 that can be changed to a diagram of K_1 by crossing changes. $\square$

From the lemma above, we see that the minimal crossing diagram is an essential condition to the definition of descendant knots. While the unknotting number is not always realized in the minimal crossing diagram, the descendant definition loses significance without the minimal crossing diagram condition, as we can get from one knot to another for all knots.

Example 2.11. We will now examine an example that illustrates the difference between the unknotting number and the ancestor-descendant relationship. Consider the following knot

[3]A **knot shadow** is the diagram of the knot such that at crossings, the overstrand and understrand are not specified.

with Conway notation 514. It is known that this knot cannot be drawn with fewer crossings, so its crossing number is 10. It is also known that this is the only diagram (up to planar isotopy and mirror reflection) of this knot with 10 crossings, [1].

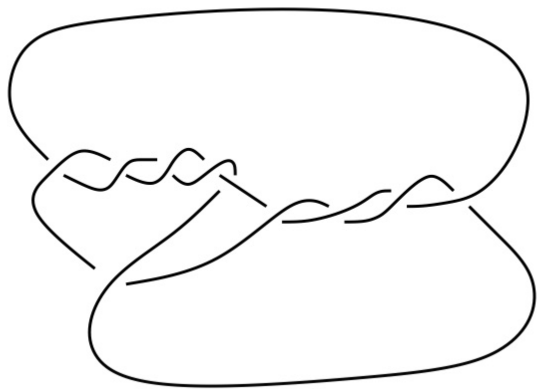


FIGURE 13. The knot 514.

It can be shown with a simple exercise that it takes at least three crossing changes in the diagram in Figure 13 to unknot this knot. Hence, by the definition of ancestor-descendant relationship, we can say that this knot is an ancestor of the unknot and that they are related by three crossing changes.

On the other hand, the unknotting number of the knot 514 is in fact 2, and the unknotting number is realized in a different diagram with more crossings. The knot 514 is an example of a knot whose unknotting number is not realized in its minimal crossing diagram.

This example illustrates the difference between the unknotting number and the ancestor-descendant relationship. The knot 514 is an ancestor to the unknot related by three crossing changes, but it can be unknotted with just two crossing changes.

In Theorem 2.6, we proved that any knot diagram can be unknotted by crossing changes. By the definition of ancestor-descendant relationship, a consequence of Theorem 2.6 is:

Theorem 2.12. *Every knot is an ancestor of the unknot.*

Based on the definition of the knot ancestor-descendant relationship, we can also prove the following lemma:

Lemma 2.13. Every smooth knot K is an ancestor of its mirror.

Proof. Given a minimal crossing diagram of K, we can change all the crossings in the diagram to obtain its mirror, $m(K)$. It follows by definition that K is an ancestor of $m(K)$. Similarly, $m(K)$ is also an ancestor of K. □

Cantarella, Henrich, Magness, O'Keefe, Perez, Rawdon, and Zimmer in [3] also showed that two key knot families, twist knots and $(2, p)$-torus knots, are distinct in that they are closely related to each other in terms of ancestor-descendant relationship and are rather insular. They provided the following results about the two families:

Theorem 2.14. *The knot K is a descendant of twist knot T_n if and only if $K = T_k$ for some integer k with $0 \leq k \leq n$.*

Theorem 2.15. *The knot K is a descendant of torus knot $T_{2,p}$ if and only if $K = T_{2,q}$ for some $0 < q \leq p$, where p and q are odd integers.*

3. Transverse Knots

In the previous section, we studied smooth knots in $\mathbb{R}^3$. Now we want to consider smooth knots that satisfy an additional condition imposed by a **contact structure**.

In multivariable calculus, it is common to study **vector fields**, where every point in space is associated with a vector and the vector varies smoothly from point to point. A **plane field** on $\mathbb{R}^3$ is given by associating a two-dimensional plane to every point in $\mathbb{R}^3$, and the planes vary smoothly. We will look at some examples of plane fields.

Example 3.1. The following is a basic example of a plane field. At each point $(x, y, z) \in \mathbb{R}^3$ we associate the plane $\eta_{(x,y,z)}$ that is parallel to the xy-plane. In other words, the plane $\eta_{(x,y,z)}$ is spanned by the vectors $\vec{i}$ and $\vec{j}$.

This is an example of an **integrable plane field**, since these planes arise as tangent planes to a partition of $\mathbb{R}^3$ into 2-dimensional surfaces. In particular, consider $\mathbb{R}^3 = \amalg L_c$ where $L_c = \{z = c\}$ is the 2-dimensional plane. For all points $(x, y, c) \in L_c$, $\eta_{(x,y,z)} = T_{(x,y,c)}L_c$.

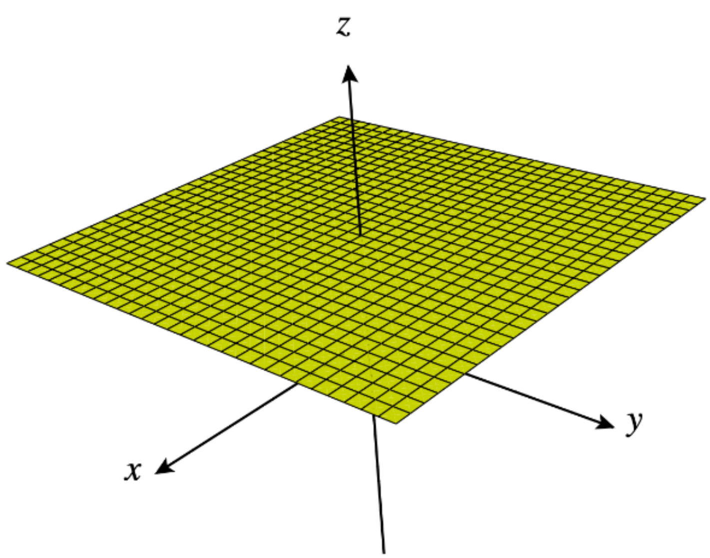


Figure 14. A tangent plane $T_{(x,y,c)}L_c$ in an integrable plane field.

More generally, if we begin by partitioning $\mathbb{R}^3$ into smooth 2-dimensional surfaces so that $\mathbb{R}^3 = \amalg S_i$, then we can obtain a plane field by considering the 2-dimensional tangent planes to these surfaces.

Definition 3.2. A plane field on $\mathbb{R}^3$ is called **integrable** if it can be seen as the tangent planes to some partition of $\mathbb{R}^3$ by two-dimensional surfaces.

Example 3.3. In contrast, we will now focus on plane fields that are **maximally non-integrable**, which means the planes are never the tangent planes for any surface. In this second example we consider the **standard contact structure**, where the plane $\xi_{(x,y,z)}$ at each point is spanned by $\vec{j}$ and $\vec{i} + y\vec{k}$, as shown in the diagram below.

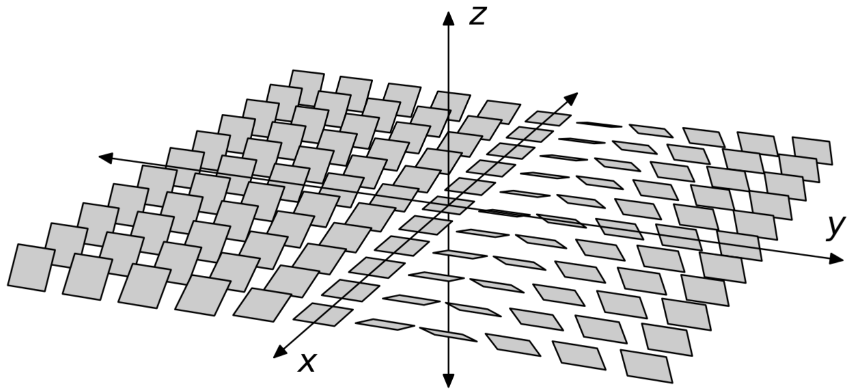


FIGURE 15. The standard contact structure on $\mathbb{R}^3$; invariant under translations in z-direction.

Observe how the slope of the planes becomes steeper as we go further down the y axis (towards the right direction), creating a sort of "twist." Because of the "twist" constantly imposed by the structure, these planes are not tangent to any surface. Hence, there does not exist a two-dimensional surface in $\mathbb{R}^3$ such that the planes are tangent to the surface.

Definition 3.4. **Transverse knots** are oriented smooth knots such that at every point in the knot, the tangent vector does not lie in the contact plane; that is, $\gamma'(t) \notin \langle \vec{j}, \vec{i} + y\vec{k} \rangle \mid_{\gamma(t)}$.

To understand what transverse means in terms of parameterization, we turn to the following lemma:

Lemma 3.5. Consider a curve $\gamma(t) = (x(t), y(t), z(t))$. If $z'(t) - y(t)x'(t) = 0$, then it follows that $\gamma'(t) \in \langle \vec{j}, \vec{i} + y\vec{k} \rangle \mid_{\gamma(t)}$.

Proof. Consider $\gamma'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}$. Assume that $z'(t) - y(t)x'(t) = 0$ for all t. By basic algebra, this implies that $z'(t) = y(t)x'(t)$ for all t.

Then,

$$\begin{aligned} \gamma'(t) &= x'(t)\vec{i} + y'(t)\vec{j} + y(t)x'(t)\vec{k} \\ &= y'(t)\vec{j} + x'(t)[\vec{i} + y(t)\vec{k}] \\ &\in \langle \vec{j}, \vec{i} + y\vec{k} \rangle \mid_{\gamma(t)} \text{ by definition.} \end{aligned}$$

□

By contrapositive, we have the following corollary to Lemma 3.5.

Corollary 3.6. If $\gamma'(t) \notin \langle \vec{j}, \vec{i} + y\vec{k} \rangle \mid_{\gamma(t)}$, then $z'(t) - y(t)x'(t) \neq 0$.

Simply put, if the knot is transverse and thus the tangent vector $\gamma'(t)$ does not lie in the contact plane, then $z'(t) - y(t)x'(t) \neq 0$ in terms of parameterization.

Definition 3.7. In this research, we focus on **positively transverse knots**. In terms of parameterization, the positive transversal condition is

$$z'(t) - y(t)x'(t) > 0.$$

The above condition implies that $z'(t) > y(t)x'(t)$. When $x'(t) > 0$, $y(t)$ is bounded above by the slope $\frac{dz}{dx}$, whereas when $x'(t) < 0$, $y(t)$ is bounded below by the slope $\frac{dz}{dx}$, [16].

It is important to note that transverse knots always have an orientation. An *orientation* is defined by choosing a direction to travel around the knot, [1]. Changing the orientation turns a positive transverse knot into a negative transverse knot and vice versa. When an orientation is assigned to a knot diagram, we can define *positive and negative crossings*.

Definition 3.8. At any crossing, when a clockwise rotation takes the understrand to overstrand, the crossing is **positive**. If a counterclockwise rotation takes the understrand to overstrand, the crossing is **negative**.

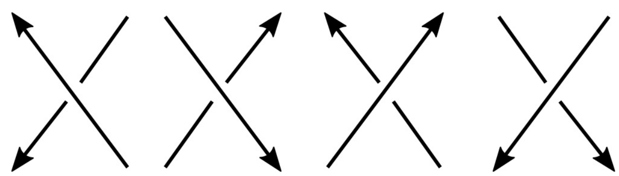


FIGURE 16. Positive crossings.

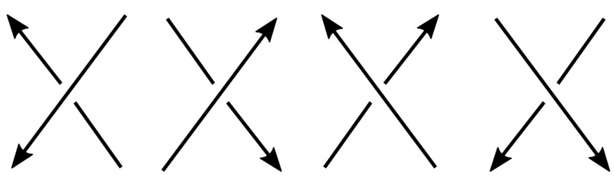


FIGURE 17. Negative crossings.

We observe transverse knots in their **front diagrams**, which are xz-diagrams in 3-dimensional space where x is on the horizontal axis and the z is on the vertical axis. To keep the standard right-handed orientation, the positive y-axis points into the page. The transversal condition creates **forbidden** vertical tangencies and crossing in front diagrams.

Lemma 3.9. In any front diagram of a transverse knot, the following vertical tangencies and crossing are forbidden:

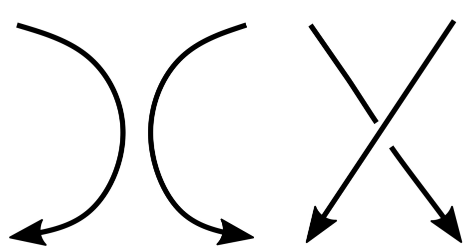


FIGURE 18. Downward vertical tangencies and down-down positive crossings are forbidden.

Proof. We will first prove that the downward vertical tangencies are forbidden.

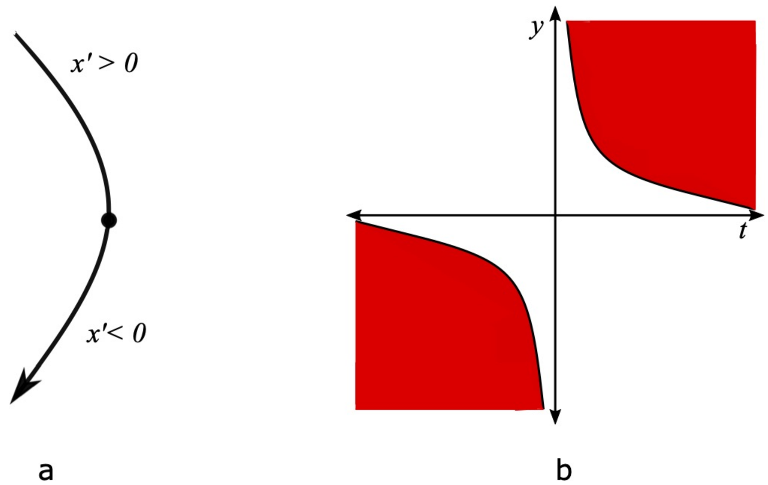


FIGURE 19. (a) A rightward downward tangency; (b) its $\frac{dz}{dx}$ plotted in a Cartesian coordinate system where t is the horizontal axis and y the vertical axis.

First consider Figure 19 (a). During the upper half of the downward vertical tangency, we have $x' > 0$ because the orientation is from left to right. Thus, by the positive transversal condition, $y < \frac{dz}{dx}$. Similarly, during the lower half of the downward vertical tangency, $x' < 0$ because the orientation is from right to left. It must be that $y > \frac{dz}{dx}$ by the positive transversal condition.

In Figure 19 (b), we have approximately graphed $y = \frac{dz}{dx}$ in a Cartesian coordinate system where t is the horizontal axis and y is the vertical axis. The colored area represents all possible y-values according to the positive transversal condition. We see that under the positive transversal condition, there exists a t for which it is impossible to define a y-value.

A similar argument follows for leftward vertical tangencies.

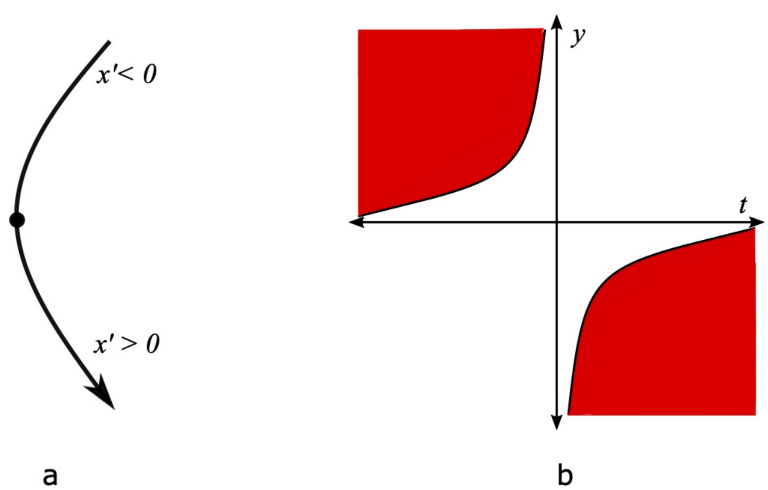


FIGURE 20. (a) A leftward downward tangency; (b) its $\frac{dz}{dx}$ plotted in a Cartesian coordinate system where t is the horizontal axis and y the vertical axis.

Thus, downward vertical tangencies are forbidden in front diagrams of a transverse knot. Now we want to prove that the down-down positive crossings are also forbidden. Consider the following diagram.

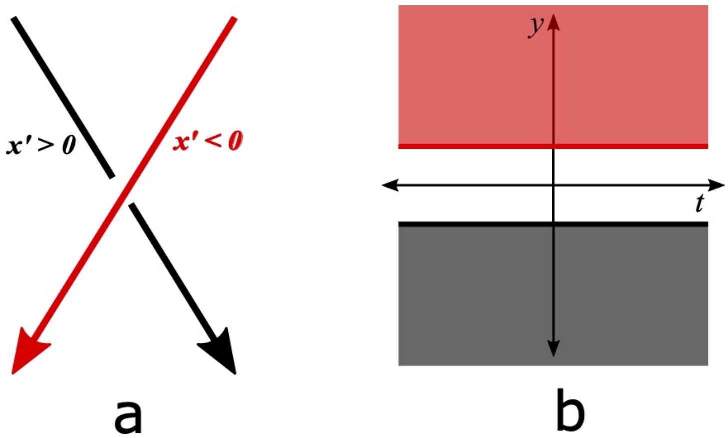


FIGURE 21. (a) A down-down positive crossing; the black strand has a greater y-coordinate. (b) $\frac{dz}{dx}$ for both the understrand and the overstrand of the crossing plotted in a Cartesian coordinate system where t is the horizontal axis and y is the vertical axis.

First consider the understrand in Figure 21 (a). Because the orientation is from left to right, we have $x' > 0$. By the positive transversal condition, it must be that $y < \frac{dz}{dx}$ for the understrand. Because the understrand has a negative slope as shown in Figure 21 (a), it follows that $\frac{dz}{dx} < 0$. Hence, we have $y < \frac{dz}{dx} < 0$ for the understrand.

Now consider the overstrand. We have $x' < 0$ because the orientation is from right to left. Hence, it must be that $y > \frac{dz}{dx}$ for the overstrand by the positive transversal condition. Because the overstrand has a positive slope, we know that $\frac{dz}{dx} > 0$. Thus, it must be that $y > \frac{dz}{dx} > 0$ for the overstrand.

This is represented in Figure 21 (b), where we have approximately graphed $y = \frac{dz}{dx}$ for both the overstrand and the understrand in a Cartesian coordinate system where t is the horizontal axis and y the vertical axis. The colored area represents all possible y-coordinates according to the positive transversal condition, the red area for the overstrand and the black area for the understrand.

But the understrand must have a y-value greater than the overstrand. Recall that we are looking at front diagrams, which are xz-diagrams in 3-dimensional space. When we think about these front diagrams in 3-dimensional space, the y-axis represents the depth and y-value increases as we go deeper into the diagram. Hence, the down-down positive crossing is forbidden in a front diagram of a transverse knot. □

We will call all other vertical tangencies and crossings that can be realized in front diagrams under the transversal condition **allowable**.

Hence, the transversal condition poses restrictions that distinguish transverse knots from smooth knots. However, we can always produce a transverse representative from any smooth knot diagram.

Lemma 3.10. Every smooth knot has a transverse representative.

Proof. Begin with any smooth knot diagram. In order to create a transverse representative, we must remove all forbidden downward vertical tangencies and forbidden down-down positive crossings. First, we remove forbidden vertical tangencies through the operation shown below in Figure 22:

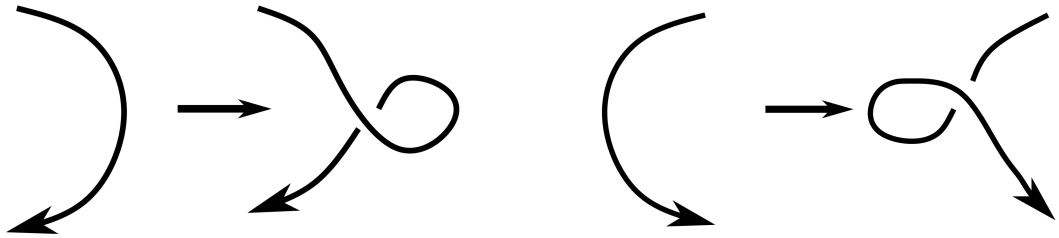


FIGURE 22. Removing forbidden vertical tangencies by introducing trivial loops.

By introducing a new trivial loop to the forbidden vertical tangency, we remove the forbidden vertical tangency without creating other forbidden segments. The new crossing of the loop is an allowable crossing, and the new vertical tangency has an upward orientation and hence is also allowable. Since this move is a Reidemeister move for smooth knots, this does not affect the topological knot type of the new diagram.

Now, we want to remove forbidden down-down positive crossings from the diagram. We use the operation shown below in Figure 23:

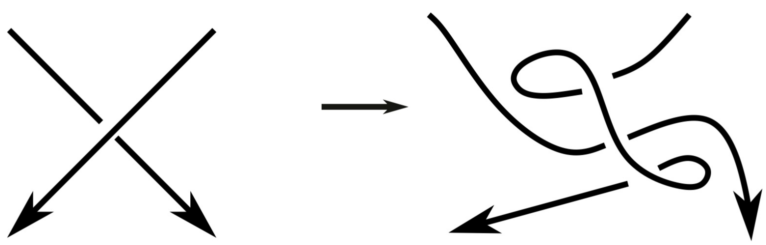


FIGURE 23. Removing forbidden crossing.

Through this move, we can alter the down-down positive crossing to an allowable crossing under the transversal condition. The operation introduces two new trivial loop crossings, but as they are both negative they are allowable. In addition, no forbidden vertical tangency is introduced. This move also does not affect the topological knot type, since it consists of isotopy and Reidemeister moves for smooth knots.

Hence, by removing forbidden vertical tangencies and forbidden crossings, we can form a transverse representative from any smooth knot. □

In the first section, we learned how to show that two smooth knots are equivalent by showing their diagrams are equivalent through planar isotopy and Reidemeister moves. Similarly, two transverse knots are equivalent if we can get from one knot to the other through *transverse isotopy* and *transverse Reidemeister moves*.

Two transverse knots T_1 and T_2 are *transversely isotopic* if we can deform T_1 to T_2 without breaking and re-gluing the strands and without introducing any forbidden segments under the transversal condition in the process of deformation. More formally:

Definition 3.11. Two transverse knots T and $\widetilde{T}$ are **transversely isotopic** if there is a family of transverse knots T_t, parameterized by $t \in [0, 1]$, such that $T_0 = T$ and $T_1 = \widetilde{T}$.

Just as there are Reidemeister moves for smooth knots, there are Reidemeister moves for transverse knots. The following in Figure 18 are the two Reidemeister moves for front diagrams of transverse knots. Orientations on the strands are allowed in all ways that do not produce forbidden vertical tangencies or forbidden crossings.

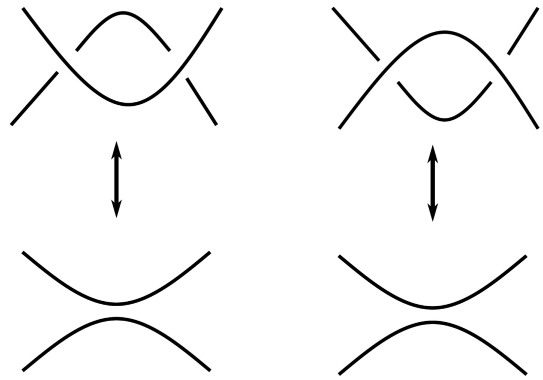


FIGURE 24. Transverse Type II Reidemeister moves.

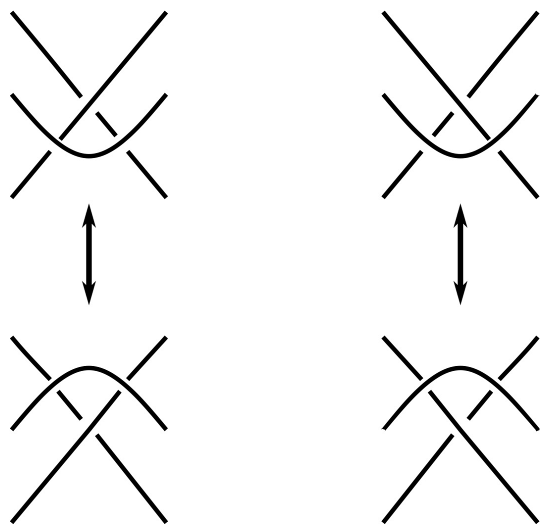


FIGURE 25. Transverse Type III Reidemeister moves.

In 1992, mathematician Jacek Swiatkowski proved in [15]:

Theorem 3.12. *Two transverse knots are equivalent up to transverse isotopy if and only if their xz-diagrams are related by the transverse Reidemeister moves in Figures 24 and 25.*

While we can use transverse isotopy and transverse Reidemeister moves to show that two transverse knots are equivalent, we can use transverse knot invariants to show that two transverse knots are different. There are two classical transverse knot invariants.

The *topological knot type* (smooth knot type) is one of the two classical invariants that distinguish different transverse knots, and the other is the *self-linking number*:

Definition 3.13. A **self-linking number** is the writhe of the knot in its front diagram.The **writhe** of a knot diagram D is defined by $w(D) = p(D) - n(D)$, where $p(D)$ is the number of positive crossings of D and $n(D)$ is the number of negative crossings of D.

An important result about transverse knots is that there is an upper bound on the self-linking number for all transverse knots, [2]. Before we state this theorem, recall that a *Seifert surface* for K is an orientable surface in $\mathbb{R}^3$ with the knot K as its one boundary component. It is known that a knot has many Seifert surfaces of different genus, and $g(K)$ is defined as the minimal genus of a Seifert surface for the knot K.

Theorem 3.14 (Bennequin)**.** *Let T be a transverse knot in $(\mathbb{R}^3, \xi_{std})$ whose smooth knot type is K_T. Then,*

$$sl(T) \leq 2g(K_T) - 1$$

where $g(K_T)$ is the minimal genus of the smooth knot type K_T.

To calculate the Bennequin bound of the self-linking number for a transverse knot, we simply refer to the genus chart in the *Knotinfo* website, [14]. Other upper bounds to $sl(T)$ have been found. An important problem is to find the $\max\{sl(T)\}$, where T ranges over transverse representatives of a fixed knot type. For some knot types, this maximum is known.

Given a transverse knot, there is a simple way to get another transverse knot in the topological knot type: *stabilization* [2, 9].

Definition 3.15. The **stabilization** of a transverse knot T is formed by taking an arc in the front diagram and doing a "double twist" as shown in Figure 26 below:

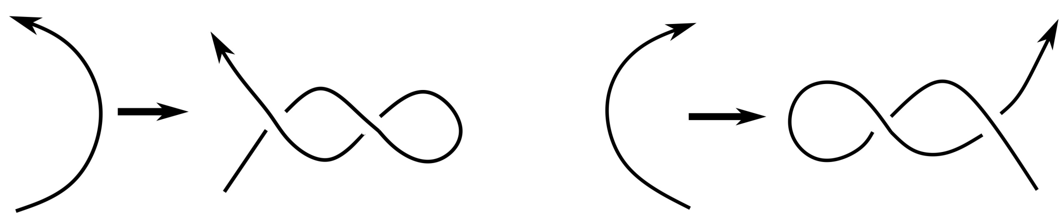


FIGURE 26. Stabilizations.

We use the notation $S^n(T)$ to express an n-th stabilization of T. Stabilization is a well-defined operation; it does not depend on at what point the stabilization is done. Stabilization also does not introduce any forbidden segment, so it does not violate the transversal condition and the newly produced knot is also a transverse knot.

Note that stabilization decreases sl by 2, as the operation introduces two new negative crossings. Since the self-linking number changes, stabilization guarantees that we produce a different transverse knot. The topological knot type remains the same, since the operation is a series of Type I Reidemeister move for smooth knots.

We saw in Lemma 3.10 that every knot has a transverse representative. In other words, for every smooth knot, there exists a transverse knot that has the smooth knot as its topological knot type. But in fact, smooth knots have many different transverse representatives. We will use the stabilization operation to prove this.

Lemma 3.16. Every smooth knot has an infinite number of non-equivalent transverse representatives.

Proof. By Lemma 3.10, we know every smooth knot has a transverse representative. Let T_1 be a transverse representative of some smooth knot K and $sl(T_1) = n$ for some integer n. To show there are many other transverse representatives, we use the stabilization operation.

Suppose we obtain T_2 by a stabilization on T_1. Then, it follows that $sl(T_2) = n - 2$ and T_1 and T_2 are different transverse knots by the definition of knot invariants. We can then do another stabilization on the new T_2 to produce T_3 with $sl(T_3) = n - 4$, and so on.

Hence, we can use stabilizations to show that there is an infinite number of non-equivalent transverse knots with the same topological knot type. □

In general, knot invariants are used to distinguish knots and do not necessarily confirm the equivalence of different diagrams. However, for transverse knots, there exist certain knot types that are completely determined by their self-linking number, [11].

Definition 3.17. A knot type is **transversely simple** if all transverse knots in this knot type are completely determined by their self-linking number.

Several knot types are known to be tranversely simple. For example, Eliashberg proved that the unknot is transversely simple, [6]. Similarly, Etnyre and Etnyre-Honda have shown that torus knots and the figure eight knot are transversely simple, [7, 8].

4. Crossing Changes in Transverse Knots

We previously defined the unknotting number and ancestor-descendant relationships for smooth knots. Now we want to explore how they can be adapted to study transverse knots.

The most important part of the unknotting number and the ancestor-descendant relationship is crossing change, whether or not we can get from one knot to another through crossing changes. In transverse knots, crossing change is limited due to a forbidden crossing. We call crossing changes that are possible in transverse knots **transverse crossing changes**. Below are the three sets of **transverse crossing changes**.

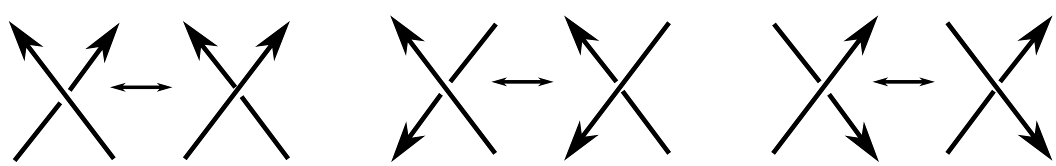


Figure 27. Transverse crossing changes.

On the other hand, the following crossing is *rigid* because changing the crossing in transverse knots will result in the forbidden crossing.

Figure 28. Rigid crossing.

Lemma 4.1. Suppose K is transverse knot. If $\widetilde{K}$ is obtained from K by transverse crossing changes, then $\widetilde{K}$ is also a transverse knot.

The lemma holds because transverse crossing changes do not introduce any forbidden segment. We also want to consider how crossing changes affect the self-linking number.

Lemma 4.2. Suppose K and $\widetilde{K}$ are both transverse knots. If we obtain $\widetilde{K}$ from K by one transverse crossing change, then the difference between $sl(K)$ and $sl(\widetilde{K})$ is exactly ± 2.

Proof. Let p be the number of positive crossings in K and n the number of negative crossings in K. By definition of self-linking number, $sl(K) = p - n$. We assume that we obtain $\widetilde{K}$ from K by one transverse crossing change. There are two cases: either we change a positive crossing to a negative or a negative crossing to a positive.

In the first case, the number of positive crossings in $\widetilde{K}$ is $(p-1)$ and the number of negative crossings is $(n+1)$. Then,

$$\begin{aligned} sl(\widetilde{K}) &= (p-1) - (n+1) \\ &= (p-n) - 2 \\ &= sl(K) - 2 \end{aligned}$$

In the second case, the number of positive crossings in $\widetilde{K}$ is $p+1$ and the number of negative crossings is $(n-1)$. Then,

$$\begin{aligned} sl(\widetilde{K}) &= (p+1) - (n-1) \\ &= (p-n) + 2 \\ &= sl(K) + 2 \end{aligned}$$

□

5. Transverse Unknotting

Having established transverse crossing changes, a natural question is: can every transverse knot be converted to a transverse unknot through transverse crossing changes? To explore this question we want to look at the following diagram of a transverse trefoil.

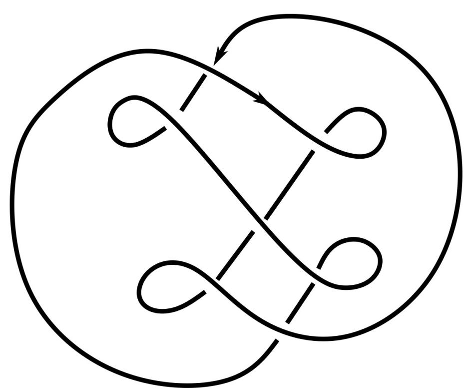


Figure 29. A transverse trefoil.

As the diagram in Figure 29 consists only of rigid crossings, we cannot make any transverse crossing change to transversely unknot it. However, there is another diagram of this transverse trefoil that can be transversely unknotted. In fact, all transverse knots can be transversely unknotted. To prove this, we first examine the theorem below:

Theorem 5.1. *Every smooth knot type has a transverse representative that can be transversely unknotted.*

Proof. By Lemma 3.10, we know that we can produce a transverse representative from any smooth knot. Begin with this transverse representative. From here we want to create another transverse representative that can be unknotted. In fact, we want to remove rigid crossings in the diagram so that transverse crossing change is always possible. However, we do not remove rigid crossings that are part of trivial loops as shown in Figure 22, since changing the crossing of a loop does not affect the unknotting process, and it is inevitable that trivial loops have been introduced to remove forbidden vertical tangencies and forbidden crossings.

Now let us examine the operation of removing all other rigid crossings; we can do this in a way similar to removing forbidden crossings:

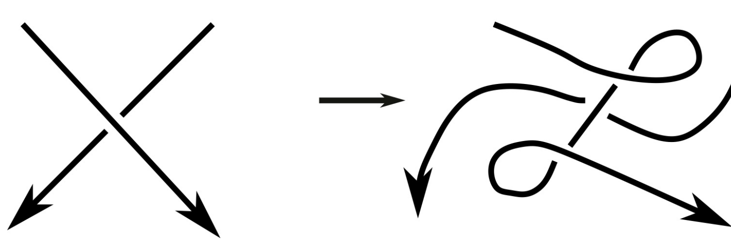


FIGURE 30. Removing rigid crossing.

The justification for this move is the same as the move for removing forbidden crossing in Figure 23. Through this move, we can adjust the rigid crossing to transform it into a non-rigid crossing. Two new trivial loops are introduced in the process to avoid forbidden vertical tangencies, and the newly formed crossings from the loops are also allowable. Similarly, this move does not affect the topological knot type.

Now we have a diagram of a transverse knot that has the original smooth knot as its topological knot type, and we can make transverse crossing changes on any non-trivial loop crossings in this diagram. Thus, we have now created a candidate transverse knot that can be unknotted. Now we will adapt the proof for Theorem 2.6 by Adams in *The Knot Book* to show that this diagram of a transverse knot can be unknotted.

Select a starting point on the knot that for convenience is not a crossing, and also select a direction to traverse the knot. Beginning at that point, we head along the knot in our chosen direction. When we come across a crossing, there are two cases: 1) it is a crossing of a trivial loop 2) it is not a crossing of a trivial loop. If it is a crossing of a trivial loop, we do not change the crossing, since changing the crossing of a trivial loop on its own does not remove the trivial loop nor influence the unknotting process. For all other crossings, the first time that we arrive at a particular crossing, we change the crossing if necessary so that the strand we are on is the overstrand. Then we continue through that crossing along the knot. If we come to a crossing that we have already been through once, we do not change that crossing, but rather continue through it on what must necessarily be the understrand. Once we return to our initial starting point, we now have a diagram of a new transverse knot that was obtained from our original transverse knot by transverse crossing changes and that will in fact be a trivial transverse knot.

To verify that this is the trivial knot in terms of topological knot type, we think of the xz diagram in the three-space, with the y axis going straight into the diagram plane. Starting at the initial point again, we place that point in three-space with y-coordinate $y = 0$. Now, as we traverse the knot, we increase the y-coordinate of each of the points on the knot. The only exception is when we reach trivial loops. The y-coordinates must be adjusted accordingly, but we can disregard the pattern in trivial loops as they do not affect the overall topological knot type. Hence, we continue to increase y-coordinate of the points until we get almost back to where we started. That last point will have y-coordinate $y = 1$. But since we gave the initial point and the last point y-coordinates $y = 0$ and $y = 1$, and these are supposed to be the same point, we had better put in a vertical bar from one to the other to complete the knot. Note then that when we look straight down the y axis at our knot, we see the diagram that we had changed the crossings to create. However, when we look at our diagram from the side, we see a diagram *with no crossings* other than the trivial loops. Hence, the topological knot type of the new transverse knot is the unknot, and the original transverse knot before the crossing changes can be transversely unknotted.

Hence, we have verified that for every smooth knot, there exists a transverse representative that can be transversely unknotted. □

Now we consider the relationship between different transverse knots with the same topological knot type. We previously defined an operation called stabilization. Fuchs and Tabachnikov proved the following about how stabilization defines relationship between between two transverse knots of the same topological knot type, [13].

Theorem 5.2. *Given two transverse knots T_1 and T_2 that are topologically isotopic, then after each has been stabilized some number of times they will be transversely isotopic.*

In Theorem 5.1, we have shown that every smooth knot has a transverse representative that can be changed to a transverse unknot by transverse crossing changes. As mentioned in Lemma 3.16, there are an infinite number of transverse unknots. In fact, if a transverse knot T can be converted to one transverse unknot by transverse crossing changes, then T can also be be converted to any transverse unknot by the following lemma:

Lemma 5.3. If K and $\tilde{K}$ are related by a stabilization, then there exists a transverse crossing change in K that produces $\tilde{K}$ and a transverse crossing change in $\tilde{K}$ that produces K.

Proof. Suppose that K and $\tilde{K}$ are related by a stabilization. For the sake of convenience, suppose that a stabilization on K produces $\tilde{K}$. We will show that there exists a transverse crossing change in $\tilde{K}$ that produces K and also that there exists a transverse crossing in K that produces $\tilde{K}$.

First, we want to show that there exists a transverse crossing change in $\tilde{K}$ that produces K. Since $\tilde{K}$ is produced by a stabilization on K, we want to show that stabilization can be undone by a transverse crossing change.

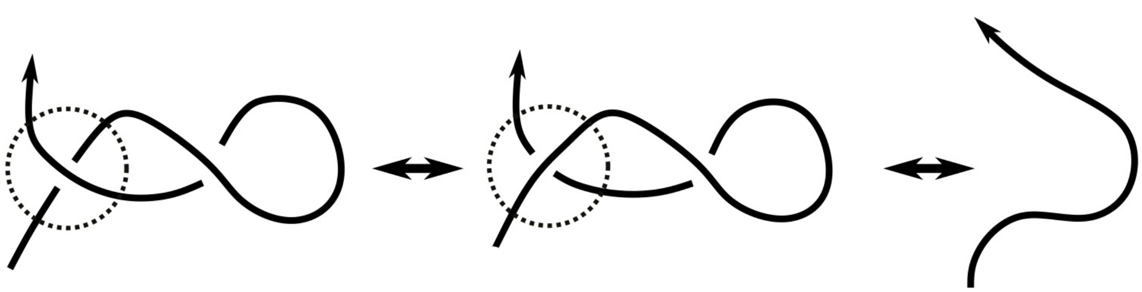


Figure 31. Crossing change in stabilization.

We move through Figure 31 from left to right. Begin with a completed stabilization on $\tilde{K}$. Then, change the marked crossing to reach the middle diagram; note that this is a transverse crossing change. Now we use transverse Type II Reidemeister move and transverse isotopy to reach the rightmost diagram, and we have undone the stabilization to produce K. Hence, we can get from $\tilde{K}$ to K through a transverse crossing change.

We also want to show that there exists a transverse crossing change in K that produces $\tilde{K}$. For this proof, we will prove that stabilization can be done by a transverse crossing change.

We now move through Figure 31 from right to left. Begin with an arc on K, as shown on the rightmost diagram. We use transverse isotopy and transverse Type II Reidemeister move to reach the center diagram in Figure 31. Observe that this transverse Type II Reidemeister move is possible because it does not introduce forbidden segments. Now, we make a transverse crossing change at the designated crossing, and we have completed a stabilization to produce $\tilde{K}$. Hence, we can get from K to $\tilde{K}$ through a transverse crossing change.

As stabilization can be done and undone through a transverse crossing change, given two transverse knots K and $\tilde{K}$ related by a stabilization, there exists a transverse crossing change that produces K from $\tilde{K}$ and a transverse crossing change that produces $\tilde{K}$ from K. ☐

Corollary 5.4. *If a transverse knot T can be converted to a transverse unknot by transverse crossing changes, then T can be converted to any transverse unknot by transverse crossing changes.*

The above corollary holds because the unknot is a transversely simple knot type. In fact, Lemma 5.3 also leads to the following corollary:

Corollary 5.5. *Given a smooth knot type K, then any two transverse knots of K are equivalently related by transverse crossing changes.*

The above corollary holds because Theorem 5.2 tells us that any two transverse knots of K are transversely isotopic after some number of stabilizations and by Lemma 5.3 we know that a stabilization can be done and undone through a transverse crossing change.

Now we prove the following theorem:

Theorem 5.6. *Every transverse knot can be transversely unknotted.*

Proof. Let T be an arbitrary transverse knot. Consider its smooth knot type, K.

By Theorem 5.1, there exists a transverse representative T' in knot type K that can be transversely unknotted. Because T and T' are topologically isotopic, by Theorem 5.2 we know that after each transverse knot has been stabilized some number of times they will be transversely isotopic, $S^n(T) = S^m(T')$.

As T' can be transversely unknotted, Lemma 5.3 implies that $S^m(T')$ can be transversely unknotted as well. Because $S^n(T) = S^m(T')$, it follows that $S^n(T)$ can be transversely unknotted. As Lemma 5.3 shows that a stabilization can be done and undone through a transverse crossing change, we conclude that T can also be transversely unknotted. ☐

6. Transverse Unknotting Number

By Corollary 5.4 and Theorem 5.6, we know that all transverse knots can be converted to any transverse unknot by transverse crossing changes. Now we want to define the transverse unknotting number. For smooth knots, the unknotting number measures the minimum number of crossing changes necessary to get to the unknot. Since there are different transverse unknots, we could define the transverse unknotting number as the minimum number of transverse crossing changes to get to any transverse unknot or a particular transverse unknot. In this paper, we choose to measure how far away we are from the transverse unknot with the maximum self-linking number, $sl = -1$. Hence, we define transverse unknotting number as the following:

Definition 6.1. A transverse knot T has **transverse unknotting number** n if there exists a diagram of the transverse knot such that transversely changing n crossings in the diagram turns T into the transverse unknot with $sl = -1$ and there is no diagram of T such that fewer transverse crossing changes would have produced the transverse unknot with $sl = -1$. We use the notation $U^{\pitchfork}_{-1}(T)$ for the transverse unknotting number of T.

Because every front diagram of a transverse knot can be turned into a front diagram of any transverse unknot by transversely changing some subset of the crossings in the diagram, it follows that every transverse knot has a finite transverse unknotting number.

Remark 6.2. If a transverse knot T can be transversely unknotted with n crossing changes, then it follow that T bounds a symplectic disk with n transverse double points. This can be shown using an argument in the proof of Lemma 6.1 in [5].

Lemma 6.3. Let T be a transverse knot that has smooth knot type S. Then,

$$\max\left\{u(S),\ \frac{sl(T)+1}{2}\ \right\} \leq U^{\pitchfork}_{-1}(T)$$

where $u(S)$ is the unknotting number of the smooth knot S and $|{}^{(sl(T)+1)}/_{2}|$ is the number of transverse crossing changes necessary to obtain $sl(T) = -1$.

Remark 6.4. Let $m = \max\{u(S), |{}^{(sl(T)+1)}/_{2}|\}$. If we can show that a transverse knot T has a diagram such that transversely changing m crossings in the diagram turns T into the transverse unknot with $sl = -1$, then $U^{\pitchfork}_{-1}(T) = m$.

Lemma 6.5. Let T be a transverse knot and consider its n-th stabilization, namely $S^n(T)$. Then, $U^{\pitchfork}_{-1}(S^n(T)) \leq U^{\pitchfork}_{-1}(T) + n$.

Proof. By Lemma 5.3, we know that we can get from $S^n(T)$ to T by n transverse crossing changes. Hence, it follows that $U^{\pitchfork}_{-1}(S^n(T)) \leq U^{\pitchfork}_{-1}(T) + n$. ☐

Corollary 6.6. For a transversely simple smooth knot type S with maximum self-linking number $m \leq -3$, suppose that the maximum self-linking number transverse representative T has transverse unknotting number $U^{\pitchfork}_{-1}(T) = |{}^{(sl(T)+1)}/_{2}|$. Then, it follows that for any transverse knot T_i in this smooth knot type S, the transverse unknotting number is

$$U^{\pitchfork}_{-1}(T_i) = \frac{sl(T_i)+1}{2}\ .$$

Proof. For a transversely simple knot type, the transverse knots are completely determined by their self-linking number and are related to each other by stabilizations. Hence, we can write every T_i as an n-th stabilization of T for some $n \in \mathbb{N}$: $T_i = S^n(T)$.

From our assumptions, $U^{\pitchfork}_{-1}(T) = |{}^{(sl(T)+1)}/_{2}|$ implies that $u(S) \leq |{}^{(sl(T)+1)}/_{2}|$ when we consider Lemma 6.3 for T. Using Lemma 6.3 for T_i, we have

$$\max\left\{u(S),\ \frac{sl(T_i)+1}{2}\ \right\} \leq U^{\pitchfork}_{-1}(T_i).$$

Because T_i is a n-th stabilization of T, we know that $sl(T_i) = sl(T) - 2n$. Then,

$$\begin{aligned}
\frac{sl(T_i)+1}{2} &= \frac{sl(T)-2n+1}{2} \\
&= \frac{sl(T)+1}{2} - n \\
&= \frac{sl(T)+1}{2} + |-n| \text{ because } sl(T) < -3 \\
&= \frac{sl(T)+1}{2} + n \text{ because } n > 0.
\end{aligned}$$

Since we know that $u(S) \leq |{}^{(sl(T)+1)}/_2|$, it follows that $u(S) < |{}^{(sl(T)+1)}/_2| + n = |{}^{(sl(T_i)+1)}/_2|$. Hence, Lemma 6.3 implies that

$$\begin{aligned}
& \max\left\{u(S),\ \frac{sl(T_i)+1}{2}\right\} = \frac{sl(T_i)+1}{2} \\
\Longrightarrow\ & \frac{sl(T_i)+1}{2} \leq U^{\pitchfork}_{-1}(T_i) \qquad (1)
\end{aligned}$$

By Lemma 6.5, $U^{\pitchfork}_{-1}(T) = |{}^{(sl(T)+1)}/_2|$ also implies that

$$\begin{aligned}
U^{\pitchfork}_{-1}(T_i) &\leq U^{\pitchfork}_{-1}(T) + n \\
&= \frac{sl(T)+1}{2} + n \text{ by our assumption} \\
&= \frac{sl(T_i)+1}{2} \quad \text{by our previous calculations.} \qquad (2)
\end{aligned}$$

Combining (1) and (2), we conclude that

$$U^{\pitchfork}_{-1}(T_i) = \frac{sl(T_i)+1}{2}\ .$$

Hence, given a transversely simple smooth knot type with maximum self-linking number $m \leq -3$, if the maximum self-linking number transverse representative has transverse unknotting number that is determined by its self-linking number, then all transverse knots of this smooth knot type have transverse unknotting number determined by their self-linking numbers as well. □

Example 6.7. We want to use some transversely simple knot types with low crossing numbers to illustrate how Corollary 6.6 can be applied.

Consider the smooth knot 5_1. Because 5_1 is a torus knot, it is transversely simple, [10]. According to the knot atlas in [4], the maximum self-linking number for 5_1 is $sl = -7$. Now we want to find the maximum self-linking number transverse representative of 5_1. Consider the diagram of T_5 in Figure 33.

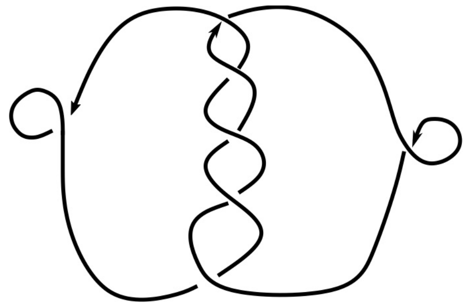


FIGURE 32. T_5, a transverse representative of 5_1.

It is not difficult to verify that T_5 is the maximum self-linking number transverse representative of 5_1. The diagram shows seven negative crossings in total, and hence $sl(T_5) = -7$. Because $u(5_1) = 2$ and $|{}^{(sl(T_5)+1)}/_{2}| = 3$, we know by Lemma 6.3 that

$$\max\left\{u(5_1),\ \frac{sl(T_5)+1}{2}\right\} = 3 \leq U_{-1}^{\pitchfork}(T_5).$$

Now we want to prove that, in fact, $U_{-1}^{\pitchfork}(T_5) = 3$ by showing that T_5 can be converted to the $sl = -1$ unknot by just 3 transverse crossing changes. Observe that all the non-trivial loop crossings of T_5 are non-rigid, so we can transversely change any of those crossings. We make the following three transverse crossing changes on T_5:

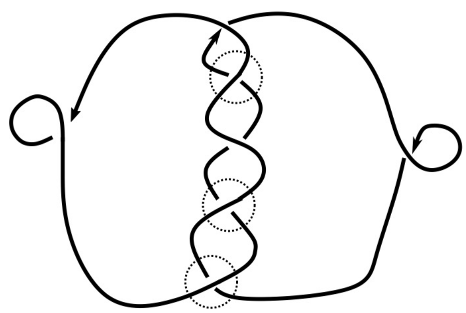


FIGURE 33. A transverse knot after three transverse crossing changes are performed on T_5. We will call this knot T_c.

Observe that now we can perform the transverse Type II Reidemeister move twice on T_c to remove two pairs of positive and negative crossings. It is then not hard to see that T_c is topologically an unknot. And because we changed three negative crossings to positive in T_5 to produce T_c, we have $sl(T_c) = sl(T_5)+6 = -7+6 = -1$. This confirms that we can convert T_5 to the $sl = -1$ transverse unknot by 3 transverse crossing changes, and $U_{-1}^{\pitchfork}(T_5) = 3$.

Using Corollary 6.6, we conclude that for any transverse knot T_i in smooth knot type 5_1, the transverse unknotting number is

$$U_{-1}^{\pitchfork}(T_i) = \frac{sl(T_i)+1}{2}\ .$$

In fact, we have verified that all smooth knots with crossing number 5 or less have a maximum self-linking number transverse representative that satisfy $U_{-1}^{\pitchfork}(T) = |{}^{(sl(T)+1)}/_{2}|$.

As all smooth knots with crossing number 5 or less are transversely simple, if the smooth knot has a maximum self-linking number $m \leq -3$, we can apply Corollary 6.6 to find the transverse unknotting number for any transverse knot in that smooth knot type. Refer to the knot atlas in [4] for information on the maximum self-linking number of knots.

7. Transverse Family Tree

Recall that for a smooth knot K_1 to be an ancestor of K_2, the knot K_2 should be obtained from a minimal crossing diagram of K_1 by some number of crossing changes. We now adapt this ancestor-descendant relationship in smooth knots to study transverse knots.

Definition 7.1. A sequence of transverse knots $(T_1, T_2, \ldots, T_n)$ is a **transverse family tree** if each T_{i+1} can be obtained from T_i by a single transverse crossing change. A transverse family is **maximal** if each T_i has maximal self-linking number in its knot type. A transverse family tree is **increasing** if the self-linking numbers of T_i are strictly increasing and **decreasing** if the self-linking numbers are strictly decreasing.

Lemma 7.2. If $(T_1, T_2, \ldots, T_n)$ is a transverse family tree, then $(T_n, \ldots, T_2, T_1)$ is a transverse family tree. Furthermore, $(T_1, T_2, \ldots, T_n)$ is maximal if and only if $(T_n, \ldots, T_2, T_1)$ is maximal. Finally, $(T_1, T_2, \ldots, T_n)$ is increasing if and only if $(T_n, \ldots, T_2, T_1)$ is decreasing, and similarly, $(T_1, T_2, \ldots, T_n)$ is decreasing if and only if $(T_n, \ldots, T_2, T_1)$ is increasing.

Proof. Assume that $(T_1, T_2, \ldots, T_n)$ is a transverse family tree. By definition, each T_{i+1} can be obtained from T_i by a single transverse crossing change. This implies that each T_i can be obtained from T_{i+1} by a single transverse crossing change too. Hence, $(T_n, \ldots, T_2, T_1)$ is also a transverse family tree.

Next, suppose that $(T_1, T_2, \ldots, T_n)$ is maximal. By definition, each T_i has maximal self-linking number in its knot type. Since $(T_1, T_2, \ldots, T_n)$ is comprised of the same T_i, it follows that $(T_1, T_2, \ldots, T_n)$ is also maximal by definition. The proof for opposite "if" direction follows in the same way.

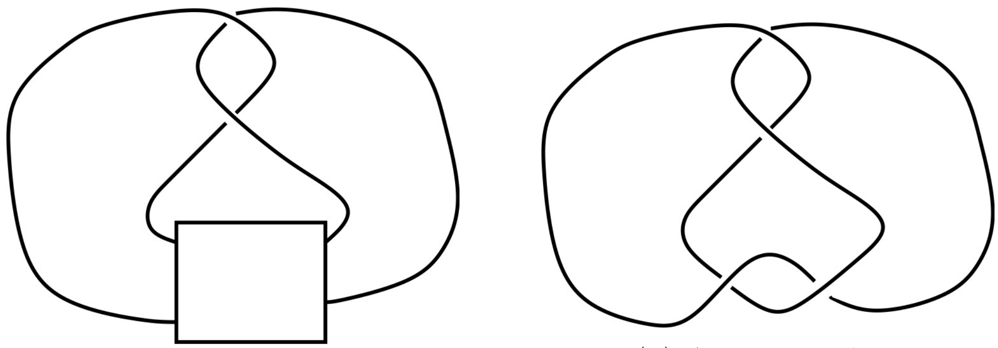
Now assume that $(T_1, T_2, \ldots, T_n)$ is increasing. By definition, the self-linking numbers of T_i are strictly increasing. Then, it follows that the self-linking numbers of T_i in $(T_n, \ldots, T_2, T_1)$ are strictly decreasing. Hence, $(T_n, \ldots, T_2, T_1)$ is decreasing. The proof for the opposite "if" direction follows in the same way. We can also prove in a similar way that $(T_1, T_2, \ldots, T_n)$ is decreasing if and only if $(T_n, \ldots, T_2, T_1)$ is increasing. $\square$

Lemma 7.3. Given any smooth knot types K_1 and K_2, there exists a transverse family tree $(T_1, \ldots, T_2)$ where T_1 is in the knot type K_1 and T_2 is in the knot type K_2. Moreover, we can assume that T_1 and T_2 have maximum self-linking numbers (but not that all knots in the tree have maximum self-linking number).

Proof. By Theorem 5.1, we know that every smooth knot has a transverse representative that can be unknotted. Take such transverse representatives of both K_1 and K_2, respectively called T_1 and T_2. We can then take the knot shadow of the composite knot $T_1 \# T_2$. The argument follows in a similar way to the proof of Lemma 2.10. We first choose the crossings on the shadow of $T_1 \# T_2$ so that we have T_1, and then transversely change crossings so that we have T_2. This process guarantees that we have a sequence of transverse knots $(T_1, \ldots, T_2)$ where each T_{i+1} can be obtained from T_i by a single transverse crossing change.

Let T_m be the maximum self-linking number transverse representative of K_1 and T_n of K_2. According to Corollary 5.5, all transverse knots in the same smooth knot type are related by transverse crossing changes.Then, we can obtain T_m from T_1 by some number of transverse crossing changes and similarly, T_n from T_2. Hence, there is exists a transverse family tree $(T_m, \ldots, T_1, \ldots, T_2, \ldots, T_n)$ where T_m is the maximum self-linking number transverse representative of K_1 and T_n of K_2. $\square$

Now we will construct some transverse family trees of twist knots. Let K_m denote a twist knot, where $m \geq 1$ is the number of right-handed half-twists, as shown in Figure 34 (A). Figure 34 (B) shows an an example of such twist knot: K_2. Observe that K_2 has two right-handed half-twists at the bottom.



(A) A diagram of K_m. (B) A diagram of K_2

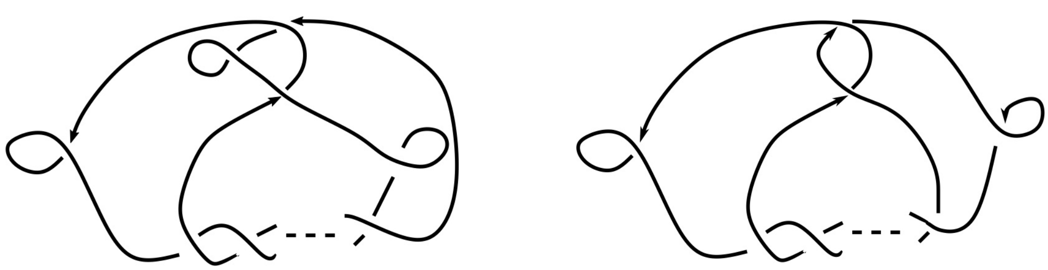
Figure 34. Twist knots.

Theorem 7.4 (Twist Knot Transverse Family Trees)**.**

(1) *For any odd $m \geq 1$, there exists a maximal decreasing transverse family tree $(T_m, T_{m-2}, \ldots, T_1)$, where T_j is a transverse representative of the twist knot K_j.*
(2) *For any even $m \geq 2$, there exists a maximal decreasing transverse family tree $(T_m, T_{m-2}, \ldots, T_2)$, where T_j is a transverse representative of the twist knot K_j.*
(3) *There exists a transverse family tree $(T_m, U, T_{m-1}, U, \ldots, T_1)$, where T_j is a transverse representative of the twist knot K_j and U is a transverse unknot.*

Before we prove Theorem 7.4, we first consider the following lemma:

Lemma 7.5. The transverse twist knots in Figure 35 have maximum self-linking number.



(A) T_e (B) T_o

Figure 35. A transverse representative of K_m (A) when m is even and (B) when m is odd. We will call each knot T_e and T_o respectively.

Proof. By simple calculation of self-linking numbers for T_e and T_o, we have $sl(T_e) = -1 - m$ and $sl(T_o) = -4 - m$. We want to confirm that these are the maximum self-linking numbers of K_m when m is even and m is odd respectively.

Consider the diagram of a general twist knot K_m in Figure 34 (A). Etnyre, Ng, and Vértesi in [12] proved the following about K_m when $m \geq 1$. In general, K_m is a transversely simple knot type when $m \geq 1$. Furthermore, the maximum self-linking number of K_m is $sl = -m-1$ when m is even and $sl = -m - 4$ when m is odd. Hence, this confirms that the diagrams Figure 35 are the maximum self-linking number transverse representative of K_m. □

Now we begin the proof of Theorem 7.4:

Proof. (1) Consider a twist knot K_m, where $m \geq 1$ is odd. Let T_m be its maximum self-linking number transverse representative. Because m is odd, T_m would look like the diagram below in Figure 36 with m twist crossings.

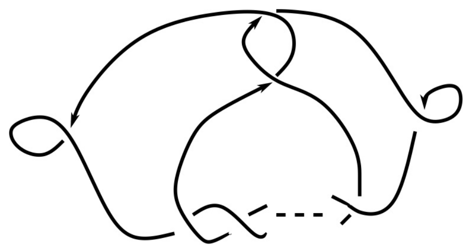
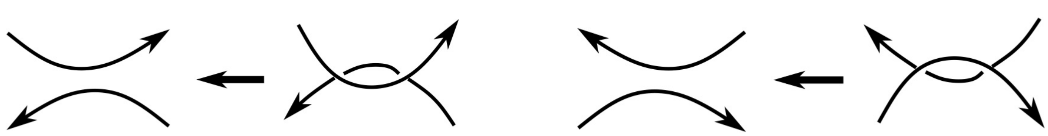


FIGURE 36. T_m where m is odd.

Now we want to show that there exists a maximal, decreasing transverse family tree $(T_m, T_{m-2}, \ldots, T_1)$ where each T_j is a maximum self-linking number transverse representative of the twist knot K_j. Observe that all twist crossings in T_m are negative crossings. For $m \geq 3$, there are $(m-1)/2$ down-down crossings in the twist of T_m. Because these crossings are rigid, there are only $(m+1)/2$ twist crossings in T_m that can be transversely changed.

Select a non-rigid twist crossing in T_m and transversely change it to a positive crossing. We then perform a transverse Type II Reidemeister move on the twist, removing one positive and one negative crossing, which leaves us with $m - 2$ twist crossings.



(A) (B)

FIGURE 37. Two valid versions of transverse Type II Reidemeister move.

We have thus produced a transverse representative of K_{m-2}, namely T_{m-2}. In fact, T_{m-2} has the maximum self-linking number, as $sl(T_{m-2}) = (-4 - m) + 2 = -4 - (m - 2)$ is the maximum self-linking number of K_{m-2} for odd $m-2$ according to Etnyre, Ng, and Vértesi's theorem in [12]. Thus, we have shown that T_m and T_{m-2} are related by a transverse crossing change, where T_m and T_{m-2} are maximum self-linking number transverse representatives of K_m and K_{m-2} respectively.

We can repeat the steps above to construct a maximal, decreasing transverse family tree $(T_m, T_{m-2}, \ldots, T_1)$ where each T_j is a transverse representative of the twist knot K_j.

(2) Consider a twist knot K_m, where $m \geq 2$ is even. Let T_m be its maximum self-linking number transverse representative. Because m is even, T_m would look like the diagram below in Figure 38 with m twist crossings.

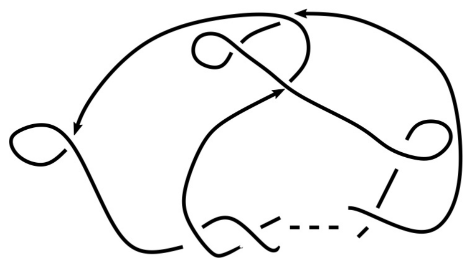
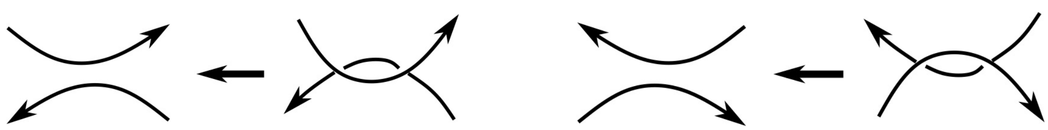


FIGURE 38. T_m where m is even.

Now we want to show that there exists a maximal, decreasing transverse family tree $(T_m, T_{m-2}, \ldots, T_2)$ where each T_j is a maximum self-linking number transverse representative of the twist knot K_j. Observe that all twist crossings in T_m are negative crossings. For $m \geq 2$, there are $m/2$ down-down crossings in the twist of T_m. Because these crossings are rigid, there are only $m/2$ twist crossings in T_m that can be transversely changed.

Select a non-rigid twist crossing in T_m and transversely change it to a positive crossing. We then perform a transverse Type II Reidemeister move on the twist, removing one positive and one negative crossing, which leaves us with $m-2$ twist crossings.



(A) (B)

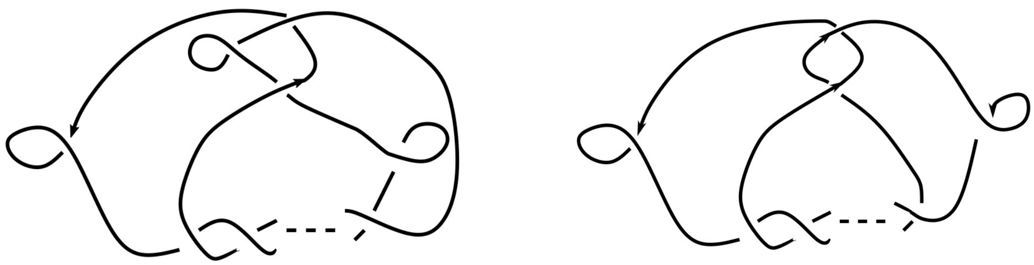
FIGURE 39. Two valid versions of transverse Type II Reidemeister move.

We have thus produced a transverse representative of K_{m-2}, namely T_{m-2}. In fact, T_{m-2} has the maximum self-linking number, as $sl(T_{m-2}) = (-1-m)+2 = -1-(m-2)$ is the maximum self-linking number of K_{m-2} for even $m-2$ according to Etnyre, Ng, and Vértesi's theorem in [12]. Hence, we have shown that T_m and T_{m-2} are related by a transverse crossing change, where T_m and T_{m-2} are maximum self-linking number transverse representatives of K_m and K_{m-2} respectively.

We can repeat the steps above to construct a maximal, decreasing transverse family tree $(T_m, T_{m-2}, \ldots, T_2)$ where each T_j is a transverse representative of the twist knot K_j.

(3) Consider a twist knot K_m, where $m \geq 1$. Let T_m be its maximum self-linking number transverse representative, either in the form of T_e or T_o in Figure 35 depending on the polarity of m. Note that for both T_e and T_o, the clasp crossings are non-rigid. We can select one clasp crossing and transversely change it, which unknots the twist knot. Hence, we have a transverse unknot, namely U. In the transverse unknot U, we then transversely change

the remaining clasp crossing from T_m. Now we have a new knot that is produced by having changed the two clasp crossings in T_m.



(A) when m is even

(B) when m is odd

FIGURE 40. New knots produced by changing two clasp crossings in T_m.

The newly produced knot is a non-alternating twist knot. In fact, it is topologically the twist knot K_{m-1}, and thus we call this new transverse knot T_{m-1}. The topological move in Figure 41 verifies this.

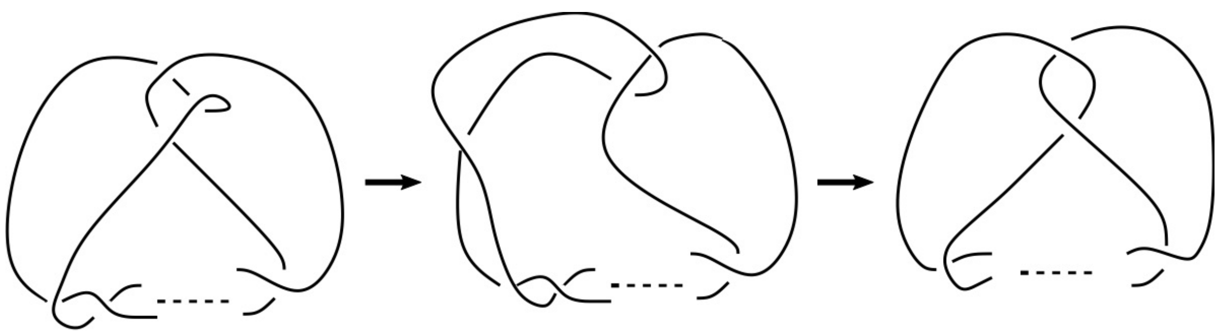


FIGURE 41. We can use this move to verify that changing the two clasp crossings from K_m topologically produces another twist knot K_{m-1}.

Now take the maximum self-linking number transverse representative of K_{m-1} that is in the form of T_e or T_o in Figure 35 depending on the polarity of $m-1$. Because K_{m-1} is transversely simple, we can stabilize the maximum self-linking number transverse representative of K_{m-1} to produce T_{m-1}. We can then repeat the process of transversely changing a clasp crossing to produce a transverse unknot and then transversely changing the remaining clasp crossing in the transverse unknot to produce T_{m-2}, and so on.

Observe that it will always be the same transverse unknot U between different transverse twist knots, as we alternate between changing positive clasp crossings to negative and changing negative clasp crossings to positive depending on the polarity of the transverse twist knot. A simple self-linking number calculation will also verify that it is the always the same transverse unknot U.

In particular, $sl(U) = sl(T_m) - 2$ if m is even and $sl(U) = sl(T_m) + 2$ if m is odd. Suppose that m is even. The clasp crossings are both positive for an even twist knot. When we change the clasp crossing to produce U, then, we turn a positive crossing to negative and hence $sl(U) = sl(T_m) - 2$. Now suppose that m is odd. The clasp crossings are both negative for an odd twist knot. When we change the clasp crossing to produce U, then, we turn a negative crossing to positive and hence $sl(U) = sl(T_m) + 2$.

Hence, we can construct a transverse family tree $(T_m, U, T_{m-1}, U, \ldots, T_1)$, where T_j is a transverse representative of the twist knot K_j and U is a transverse unknot. □

Now we will construct some transverse family trees of torus knots. In general, a (m, n)-torus knot is obtained by winding a string n times around a circle in the interior of the torus and m times around its axis of rotational symmetry, where m and n are relatively prime. A (m, n)-torus knot is equivalent to a (n, m)-torus knot. In this section, we use $K_{m,n}$ to denote a (m, n)-torus knot and have the strands form a right-handed twist for $m, n > 0$.

Theorem 7.6 ($(2, p)$-Torus Knot Transverse Family Trees)**.**

(1) *For all odd $p \geq 3$, there exists a maximal decreasing transverse family tree* $(T_{2,p}, T_{2,p-2}, \ldots, T_{2,3})$*, where $T_{2,j}$ is a transverse representative of the torus knot $K_{2,j}$.*
(2) *For all odd $n \leq -3$, there exists a maximal increasing transverse family tree* $(T_{2,n}, T_{2,n+2}, \ldots, T_{2,-3})$*, where $T_{2,j}$ is a transverse representative of torus knot $K_{2,j}$.*

Before we prove Theorem 7.6, we first consider the following lemma:

Lemma 7.7. The transverse torus knots in Figure 42 have maximum self-linking number.

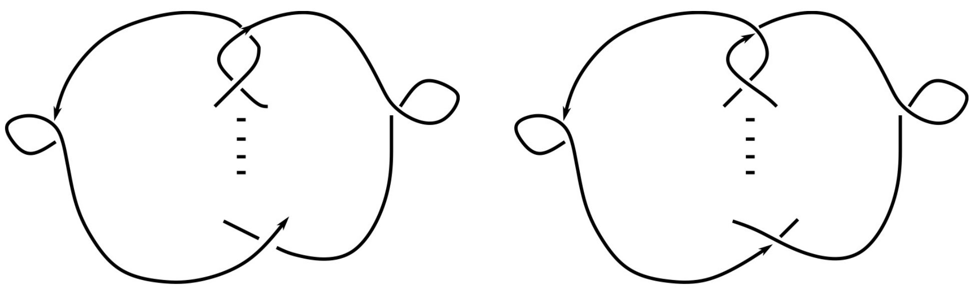


FIGURE 42. A transverse representative of a torus knot $K_{2,p}$ where p is odd, (a) when p is positive and (b) when p is negative. We will call each knot T_+ and T_- respectively.

Proof. By simple calculation of self-linking numbers for T_+ and T_-, we have $sl(T_+) = -2+p$ and $sl(T_-) = -2+p$. We want to confirm that this really is the maximum self-linking number of $K_{2,p}$ when p is positive and negative respectively.

According to Etnyre and Honda in [10], torus knots are transversely simple and the maximum self-linking number of a torus knot type is $sl(K_{m,n}) = mn - m - n$. Applying this to $K_{2,p}$, we have that $sl(K_{2,p}) = 2p - 2 - p = p - 2 = -2 + p$. Hence, this confirms that T_+ and T_- in Figure 42 are the maximum self-linking number transverse representatives of $K_{2,p}$. □

Now we begin the proof of Theorem 7.6.

Proof. (1) Consider a torus knot $K_{2,p}$ where $p \geq 3$ is odd. Let $T_{2,p}$ be its maximal self-linking number transverse representative. Because p is positive, $T_{2,p}$ would look like the diagram below in Figure 43 with p positive crossings in the middle.

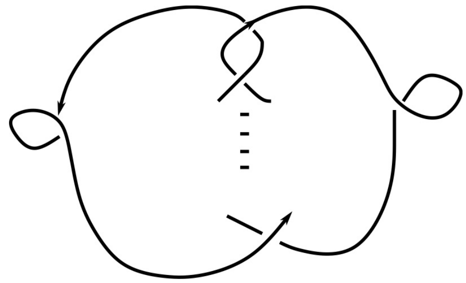


FIGURE 43. $T_{2,p}$ where p is positive.

Now we will construct a maximal, decreasing transverse family tree $(T_{2,p}, T_{2,p-2}, \ldots, T_{2,3})$, where each $T_{2,j}$ is a transverse representative of the torus knot $K_{2,j}$.

Observe that all non-trivial loop crossings in the middle of $T_{2,p}$ are non-rigid and positive, so we can make transverse crossing changes on them. Select any crossing from the middle of $T_{2,p}$ and transversely change the crossing from positive to negative. This enables to perform a transverse Type II Reidemeister move to remove a pair of crossings, one positive and one negative.

(A) (B)

FIGURE 44. Two valid versions of transverse Type II Reidemeister move.

This leaves us with $p-2$ crossings in the middle. Observe that the newly produced knot is still in the form of T_+ in Figure 42 with $p-2$ positive crossings in the middle. This implies that the newly produced transverse knot is $T_{2,p-2}$, a transverse representative of $K_{2,p-2}$. We want to verify that $T_{2,p-2}$ has the maximum self-linking number. Our previous transverse crossing change implies that $sl(T_{2,p-2}) = (-2+p)-2 = -2+(p-2)$, which is the maximum self-linking number of $K_{2,p-2}$ according to Etnyre and Honda in [10]. Hence, we have shown that $T_{2,p}$ and $T_{2,p-2}$ are related by a transverse crossing change, where $T_{2,p}$ and $T_{2,p-2}$ are maximum self-linking number transverse representatives of $K_{2,p}$ and $K_{2,p-2}$ respectively.

We can repeat the steps above to produce a series of maximum self-linking number transverse torus knots, and the self-linking numbers are strictly decreasing by 2. Hence, we can construct a maximal, decreasing transverse family tree $(T_{2,p}, T_{2,p-2}, \ldots, T_{2,3})$, where each $T_{2,j}$ is a transverse representative of the torus knot $K_{2,j}$.

(2) Consider a torus knot $K_{2,n}$ where $n \leq -3$ is odd. Let $T_{2,n}$ be its maximal self-linking number transverse representative. Because n is negative, $T_{2,n}$ would look like the diagram below in Figure 45 with $|n|$ negative crossings in the middle.

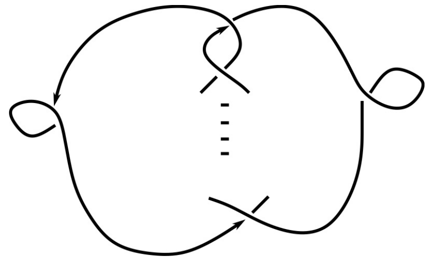


FIGURE 45. $T_{2,n}$ where n is negative.

We want to construct a maximal, increasing transverse family tree $(T_{2,n}, T_{2,n+2}, \ldots, T_{2,-3})$, where each $T_{2,j}$ is a transverse representative of the torus knot $K_{2,j}$.

Observe that all non-trivial loop crossings in the middle of $T_{2,n}$ are non-rigid and negative, so we can make transverse crossing changes on them. Select any crossing from the middle of $T_{2,n}$ and transversely change the crossing from negative to positive. This enables to perform a transverse Type II Reidemeister move as shown in Figure 44 to remove a pair of crossings, one positive and one negative.

This leaves us with $|n+2|$ negative crossings in the middle of the knot. Observe that the newly produced knot is still in the form of T_- in Figure 42 with $|n+2|$ negative crossings in the middle. This implies that the newly produced transverse knot is $T_{2,n+2}$, a transverse representative of $K_{2,n+2}$. We now want to verify that $T_{2,n+2}$ has the maximum self-linking number. Our previous transverse crossing change implies that $sl(T_{2,n+2}) = (-2+n)+2 = -2+(n+2)$, which is the maximum self-linking number of $K_{2,n+2}$ according to Etnyre and Honda in [10]. Hence, we have shown that $T_{2,n}$ and $T_{2,n+2}$ are related by a transverse crossing change, where $T_{2,n}$ and $T_{2,n+2}$ are maximum self-linking number transverse representatives of $K_{2,n}$ and $K_{2,n+2}$ respectively.

We can repeat the steps above to produce a series of maximum self-linking number transverse torus knots, and the self-linking numbers are strictly increasing by 2. Hence, we can construct a maximal, increasing transverse family tree $(T_{2,n}, T_{2,n+2}, \ldots, T_{2,-3})$, where each $T_{2,j}$ is a transverse representative of the torus knot $K_{2,j}$. $\square$

Corollary 7.8. For all odd p_1, p_2, there exists a transverse family tree $(T_{2,p_1}, \ldots, T_{2,p_2})$. If p_1 and p_2 have the same sign, there exists a maximal, monotonic transverse tree. If p_1 and p_2 have opposite signs, then there exists a monotonic transverse family tree that contains a non-maximum self-linking number transverse unknot, but all others in the family tree are maximum self-linking number transverse representatives.

Proof. By Theorem 7.6, we know that if p_1 and p_2 have the same sign, there exists a maximal, monotonic transverse family tree that contains only transverse torus knots.

Suppose that p_1 and p_2 have opposite signs. For the sake of convenience, we assume that p_1 is positive and p_2 is negative. Then, there exists a maximal transverse family tree $(T_{2,p_1}, T_{2,p_1-2} \ldots, T_{2,3})$ and $(T_{2,p_2}, T_{2,p_2+2}, \ldots, T_{2,-3})$ by Theorem 7.6. Then, it suffices to show that there is a monotonic transverse family tree between $T_{2,3}$ and $T_{2,-3}$ that contains a non-maximum self-linking number transverse unknot to complete the proof.

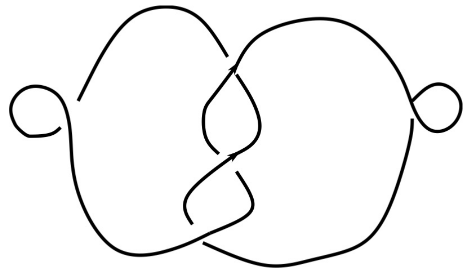


FIGURE 46. $T_{2,3}$.

Start with a diagram of $T_{2,3}$. As $T_{2,3}$ has the maximum self-linking number $sl(T_{2,3}) = 1$, we have the diagram in Figure 46. Transversely change one of the non-trivial loop crossings in the middle of $T_{2,3}$, turning a positive crossing into negative. We can now perform a transverse Type II Reidemeister move as shown in Figure 44 to remove a pair of crossings, one positive and one negative, resulting in a diagram in Figure 47.

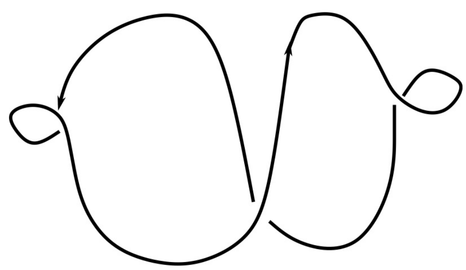


FIGURE 47. $T_{2,1}$.

We name this new transverse knot $T_{2,1}$. In fact, $T_{2,1}$ is topologically an unknot, and we have $sl(T_{2,1}) = 1 - 2 = -1$ as we changed a positive crossing to a negative crossing in $T_{2,3}$. This is the maximum self-linking number of the unknot. We then transversely change the remaining positive crossing, and now we have $T_{2,-1}$ as shown below in Figure 48.

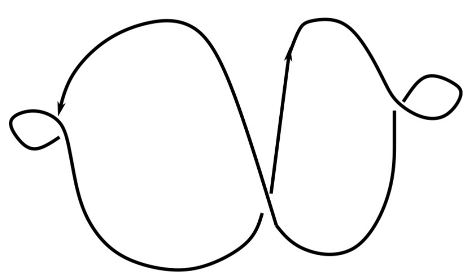


FIGURE 48. $T_{2,-1}$.

This is also topologically an unknot, but now we have $sl(T_{2,-1}) = -1 - 2 = -3$. We can then perform a transverse Type II Reidemeister move as shown in Figure 49 to add a pair of crossings, one positive and one negative.

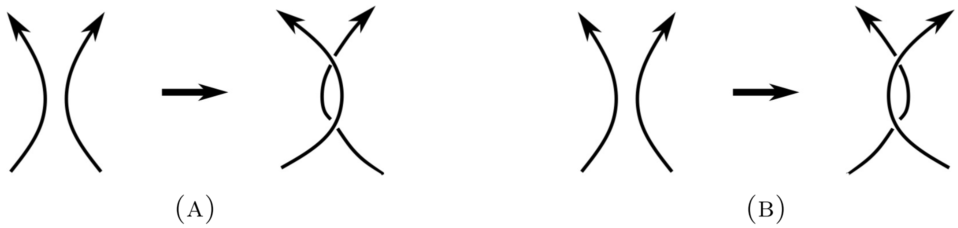


Figure 49. Two valid versions of transverse Type II Reidemeister move.

Finally, transversely change the newly added positive crossing to negative, so that we produce $T_{2,-3}$, as shown below in Figure 50, and $sl(T_{2,-3}) = -3 - 2 = -5$, as desired.

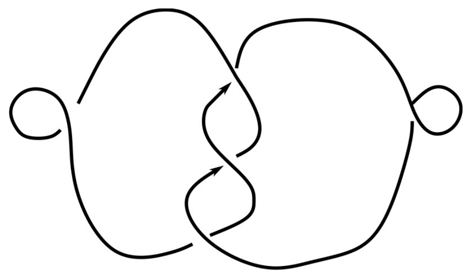


Figure 50. $T_{2,-3}$.

From the steps above, we have created a transverse family tree $(T_{2,3}, T_{2,1}, T_{2,-1}, T_{2,-3})$ where $T_{2,1}$ is a transverse unknot with $sl = -1$ and $T_{2,-1}$ is a transverse unknot with $sl = -3$. Observe that this process, similar to the one in proof of Theorem 7.6, guarantees that the transverse twist knots $T_{2,3}$ and $T_{2,-3}$ have the maximum self-linking number. Note also that this new transverse family tree is decreasing.

Using Lemma 7.2, we can now rearrange and combine the maximal transverse family trees $(T_{2,p_1}, T_{2,p_1-2} \dots, T_{2,3})$ and $(T_{2,p_2}, T_{2,p_2+2}, \dots, T_{2,-3})$ with this new transverse family tree $(T_{2,3}, T_{2,1}, T_{2,-1}, T_{2,-3})$. Hence, we have $(T_{2,p_2}, \dots, T_{2,-3}, T_{2,-1}, T_{2,1}, T_{2,3}, \dots, T_{2,p_1})$, and this transverse family tree is increasing. Observe also that all transverse knots except the transverse unknot $T_{2,-1}$ have maximum self-linking numbers. Hence, if p_1 and p_2 have opposite signs, then there exists a monotonic transverse family tree that contains a non-maximum self-linking number transverse unknot, but all others in the transverse family tree are maximum self-linking number transverse representatives.

Combing the cases when p_1 and p_2 have same signs and when p_1 and p_2 have different signs, we conclude that for all odd p_1, p_2, there exists a transverse family tree $(T_{2,p_1}, \dots, T_{2,p_2})$. $\square$

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